Bayesian Games and Mechanism Design

Definition of Bayes Equilibrium

Harsanyi [1967]

- What happens when players do not know one another’s payoffs?
- Games of “incomplete information” versus games of “imperfect information”
- Harsanyi’s notion of “types” encapsulating “private information”
- Nature moves first and assigns each player a type; player’s know their own types but not their opponents’ types
- Players do have a common prior belief about opponents’ types

Bayesian Games

There are a finite number of types $\theta_i \in \Theta_i$

There is a common prior $p(\theta)$ shared by all players

$p(\theta_{-i}|\theta_i)$ is the conditional probability a player places on opponents' types given his own type

The *stage* game has finite action spaces $a_i \in A_i$ and has utility function $u^i(a, \theta)$

Bayesian Equilibrium

A *Bayesian Equilibrium* is a Nash equilibrium of the game in which the strategies are maps from types $s_i : \Theta_i \to A_i$ to stage game actions

A_i

This is equivalent to each player having a strategy as a function of his type $s_i(\theta_i)$ that maximizes conditional on his own type θ_i (for each type that has positive probability)

$$\max_{s_i} \sum_{\theta_{-i}} u_i(s_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) p(\theta_{-i} | \theta_i)$$

Cournot Model with Types

- A duopoly with demand given by $p = 17 - x$
- A firm's type is its cost, known only to that firm: each firm has a 50-50 chance of cost constant marginal cost 1 or 3.

profits of a representative firm

$$\pi_i(c_i, x) = \left[17 - c_i - (x_i + x_{-i})\right] x_i$$

Let us look for the symmetric pure strategy equilibrium

Finding the Bayes-Nash Equilibrium

x^1, x^3 will be the output chosen in response to cost

$$\pi_i(x_i, c_i) = .5\left[17 - c_i - (x_i + x^1)\right] x_i$$
$$+ .5\left[17 - c_i - (x_i + x^3)\right] x_i$$

maximize with respect to x_i and solve to find

$x^1 = 11/2$, $x^3 = 9/2$

Industry Output

probability ¼ 11

probability ½ 10

probability ¼ 9

Suppose by contrast costs are known

If both costs are 1 then competitive output is 16 and Cournot output is 2/3rds this amount 10 2/3

If both costs are 3 then competitive output is 14 and Cournot output is 9 1/3

If one cost is 1 and one cost is 3 Cournot output is 10

With known costs, mean industry output is the same as with private costs, but there is less variation in output

Sequentiality

Kreps-Wilson [1982]

Subforms

Beliefs: *assessment* a_i for player i probability distribution over nodes at each of his information sets; *belief* for player i is a pair $b_i = (a_i, \pi^i_{-i})$ consisting of i's assessment over nodes a_i, and i's expectations of opponents' strategies $\pi^i_{-i} = (\pi^i_j)_{j \neq i}$

Beliefs come from strictly positive perturbations of strategies

belief $b_i \equiv (a_i, \pi^i_{-i})$ is *consistent* (Kreps and Wilson) if where a_i^n $a_i = \lim_{n\to\infty} a_i^n$ obtained using Bayes rule on a sequence of strictly positive strategy profiles of the opponents, $\pi^{i,m}_{-i} \to \pi_{-i}$

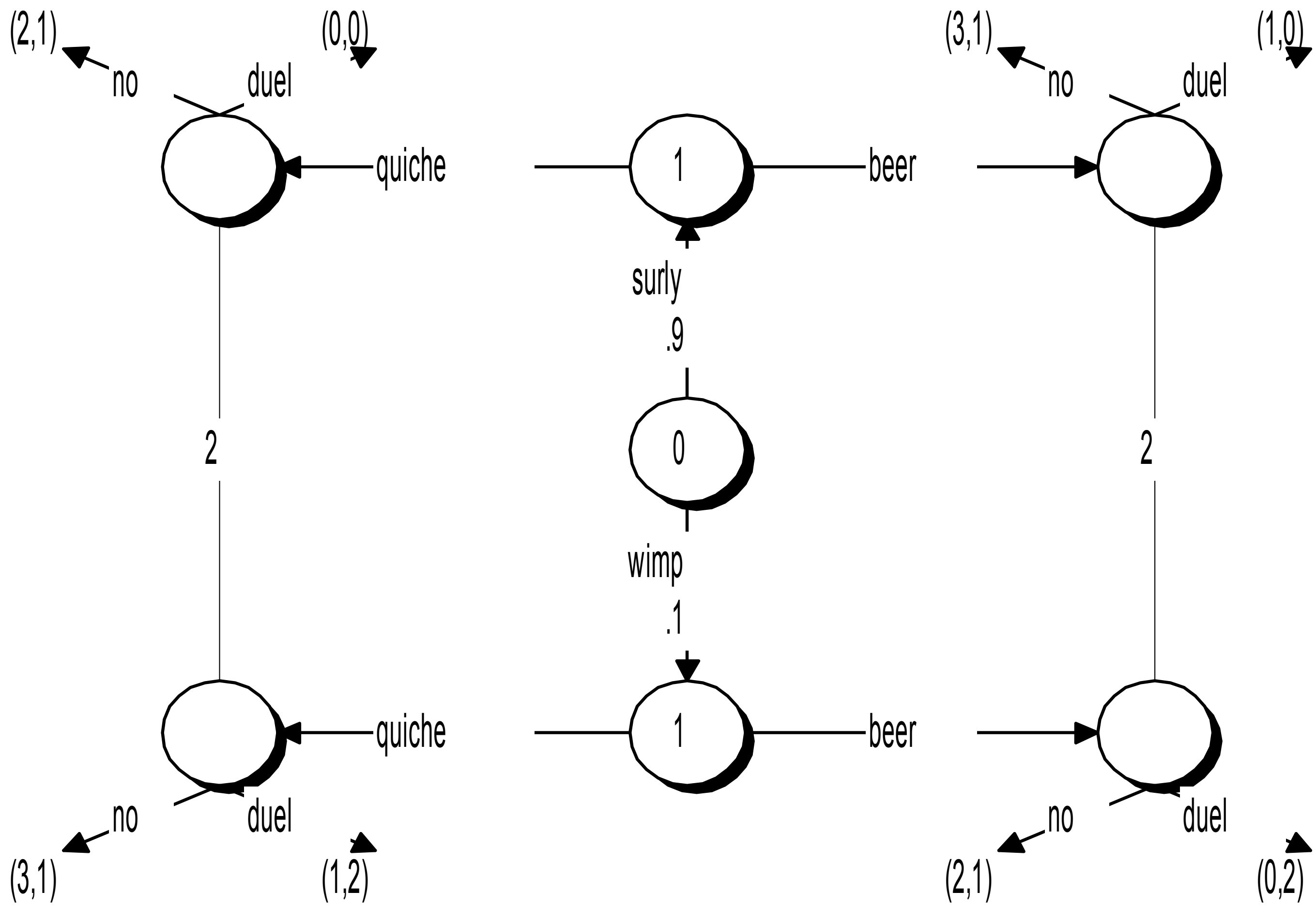
Sequential Optimality

given beliefs we have a well-defined decision problem at each information set; can define optimality at each information set

A sequential equilibrium is a behavior strategy profile π and an assessment a_i for each player such that (a_i, π^i_{-i}) is consistent and each player optimizes at each information set

Signaling

Cho-Kreps [1987]



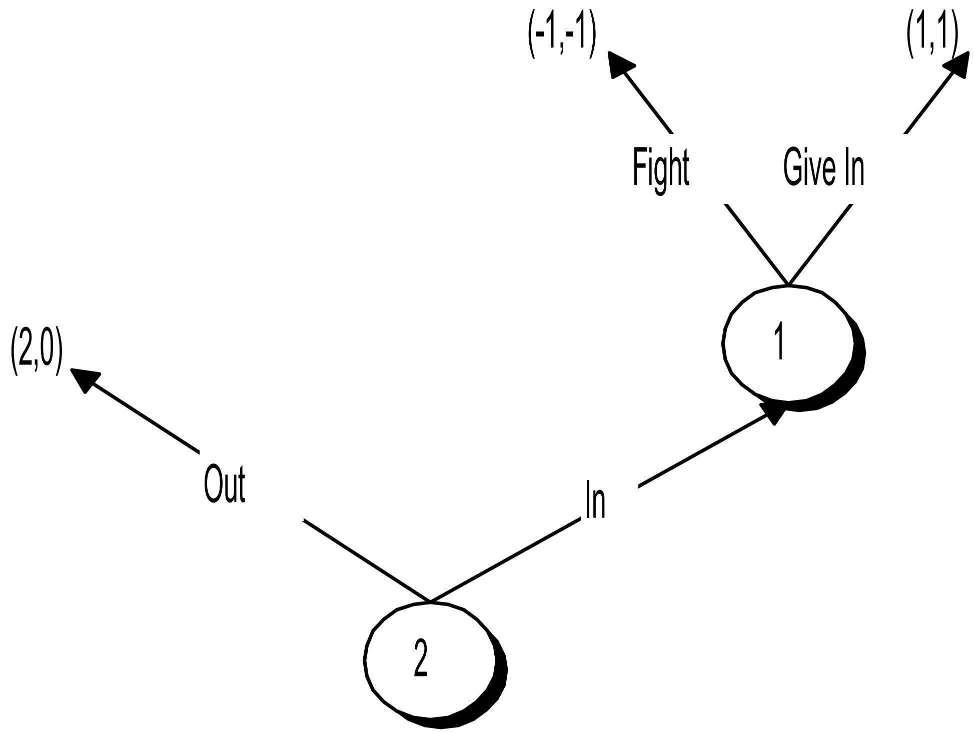
Types of Equilibrium

sequential vs. trembling hand perfect

pooling and separating

Chain Store Paradox

Kreps-Wilson [1982], Milgrom-Roberts [1982]



finitely repeated model with long-run versus short-run

Reputational Model

two types of long-run player $\omega \in \Omega$

“rational type” and “committed type”

“committed type” will fight no matter what

types are privately known to long-run player, not known to short run player

Kreps-Wilson; Milgrom-Roberts

Solve for the sequential equilibrium; show that at the time-horizon grows long we get no entry until near the end of the game

“triumph of sequentiality”

The Holdup Problem

- Chari-Jones, the pollution problem
- problem of too many small monopolies

ρ is the profit generated by an invention with a monopoly with a patent, drawn from a uniform distribution on $[0, 1]$, private to the inventor

φ^F is the fraction of this profit that can be earned without a patent

To create the invention requires as input N other existing inventions

It costs ϵ/N to make copies of each of these other inventions, where $\epsilon < 1/2$ and $\epsilon/\varphi^F < 1$

Case 1: Competition

if $\varphi^F \rho \geq \epsilon$ the new invention is created, probability is $1 - \epsilon/\varphi^F$.

Case 2: Patent

Each owner of the existing inventions must decide a price p_i at which to license their invention; φN current inventions are still under patent

Subgame Perfection/Sequentiality implies that the new invention is created when $\varphi\rho \geq \sum_i p_i$

Profit of a preexisting owner $(1 - \frac{(\varphi N - 1)p + p_i}{\varphi})p_i$

FOC $1 - \frac{(\varphi N - 1)p + 2p_i}{\varphi} = 0$

unique symmetric equilibrium $p = \varphi/(\varphi N + 1)$;
$\sum_i p_i/\varphi = \varphi N p/\varphi$

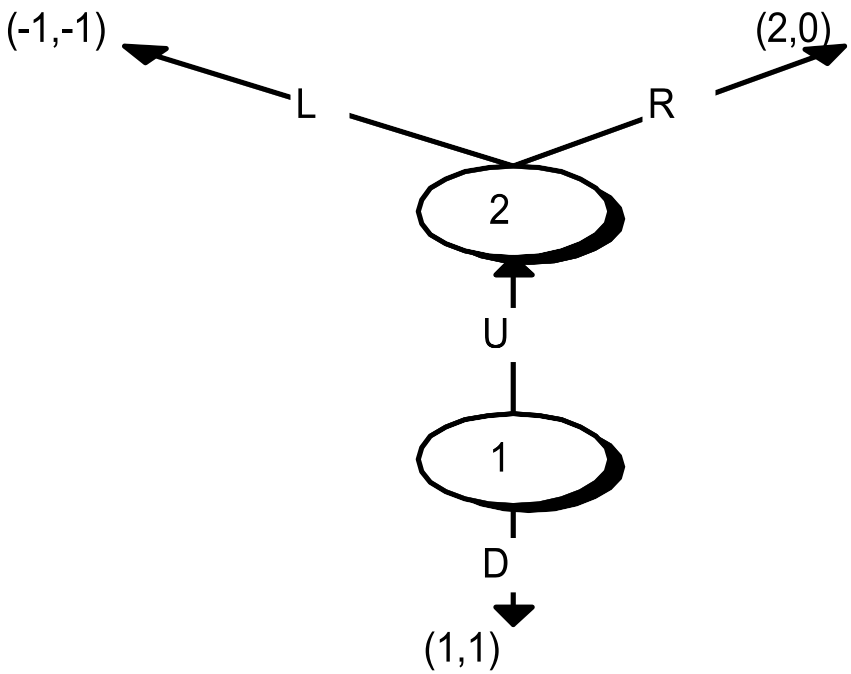
corresponding probability of invention is $1/(\varphi N + 1)$

Robustness

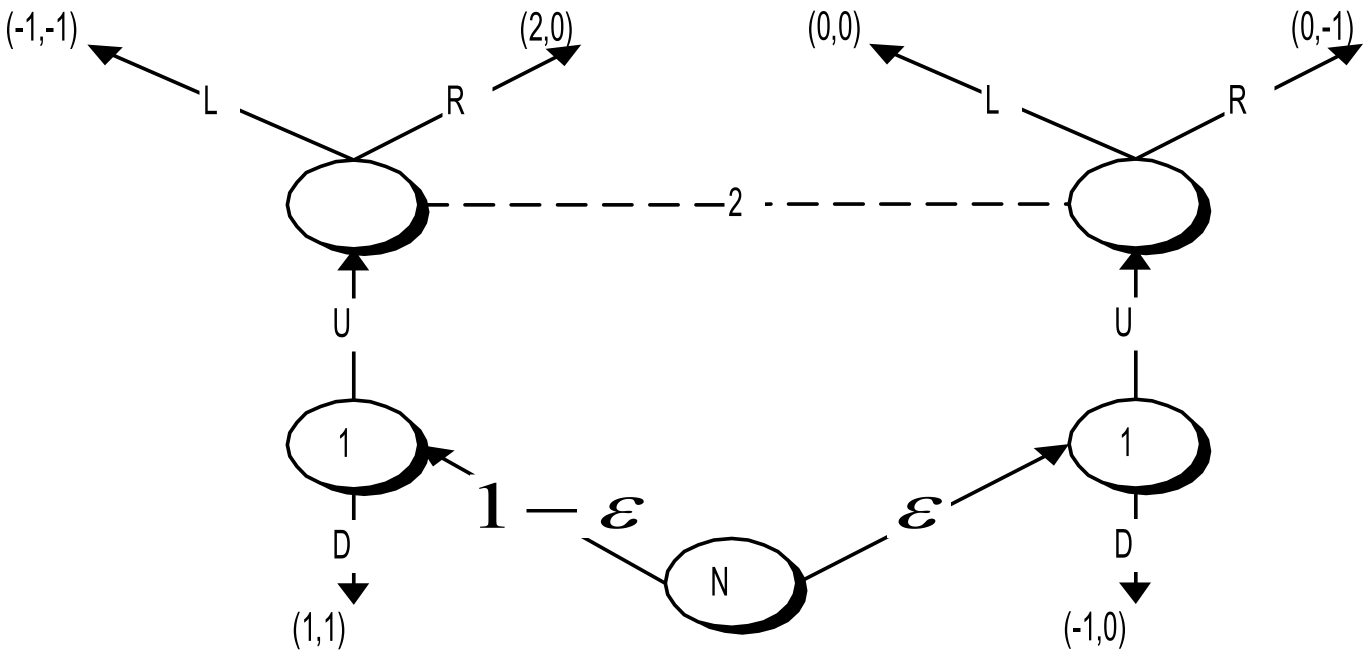
genericity in normal form games

example of Selten extensive form game

Fudenberg, Kreps, Levine [1988]



elaborated Selten game



normal form of elaborated Selten game

	L	R
$D_L D_R$	$1-2\varepsilon, 1-\varepsilon$	$1-2\varepsilon, 1-\varepsilon$
$D_L U_R$	$1-\varepsilon, 1-\varepsilon$ **	$1-\varepsilon, 1-2\varepsilon$
$U_L D_R$	$-1, -1+\varepsilon$	$2-3\varepsilon, 0$
$U_L U_R$	$-1+\varepsilon, -1+\varepsilon$	$2-2\varepsilon, -\varepsilon$

Mechanism Design: An “auction” problem

- Single seller has a single item
- Seller does not value item
- Two buyers with independent valuations

$0 \leq v^l < v^h$ low and high valuations

$\pi^l + \pi^h = 1$ probabilities of low and high valuations

what is the best way to sell the object

- Auction
- Fixed price
- Other

The Revelation Principle

Design a game for the buyers to play

- Auction game
- Poker game
- Etc.

Design the game so that there is a Nash equilibrium that yields highest possible revenue to the seller

The revelation principle says that it is enough to consider a special game

- strategies are "announcements" of types
- the game has a "truthful revelation" equilibrium

In the Auction Environment

Fudenberg and Tirole section 7.1.2

q^l, q^h probability of getting item when low and high

p^l, p^h expected payment when low and high *individual rationality constraint*

(IR) $$q^i v^i - p^i \geq 0$$

- if you announce truthfully, you get at least the utility from not playing the game

incentive compatibility constraint

(IC) $$q^i v^i - p^i \geq q^{-i} v^i - p^{-i}$$

- you gain no benefit from lying about your type

the incentive compatibility constraint is the key to equilibrium

Other constraints

q^l, q^h probability of getting item when low and high

they can't be anything at all:

probability constraints

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i/2$

(win against other type, 50% chance of winning against self)

(2) $\pi^l q^l + \pi^h q^h \leq 1/2$

(probability of getting the good before knowing type less than 50%)

Seller Problem

Maximize seller utility $U = \pi^l p^l + \pi^h p^h$

Subject to IC and IR

To solve the problem we make a guess:

IR binds for low value

$$q^l v^l - p^l = 0$$

IC binds for high value

$$q^h v^h - p^h = q^l v^h - p^l$$

The solution

$p^l = q^l v^l$ from low IR

substitute into high IC

$p^h = (q^h - q^l)v^h + q^l v^l$

plug into utility of seller

$U = \pi^l q^l v^l + \pi^h \left((q^h - q^l)v^h + q^l v^l\right)$

$U = q^l(\pi^l v^l - \pi^h v^h + \pi^h v^l) + \pi^h q^h v^h$

$\pi^l + \pi^h = 1$ so

$U = q^l(v^l - \pi^h v^h) + \pi^h q^h v^h$

Case 1: $v^l > \pi^h v^h$

$$U = q^l(v^l - \pi^h v^h) + \pi^h q^h v^h$$

(1) $0 \le q^i \le \pi^{-i} + \pi^i/2$

(2) $\pi^l q^l + \pi^h q^h \le 1/2$

Make q^l, q^h large as possible so $\pi^l q^l + \pi^h q^h = 1/2$

$$U = \frac{1/2 - \pi^h q^h}{\pi^l}(v^l - \pi^h v^h) + \pi^h q^h v^h$$

$$U = \frac{1}{2\pi^l}(v^l - \pi^h v^h) + q^h \frac{\pi^h}{\pi^l}(v^h - v^l)$$

Finish of Case 1

so q^h should be as large as possible

$$q^h = \pi^l + \pi^h/2$$

plug back into (2) to find

$$q^l = \pi^l/2$$

expected payments

$$p^l = q^l v^l,\ p^h = (q^h - q^l)v^h + q^l v^l$$

$$p^l = v^l \pi^l/2,\ p^h = v^h/2 + \pi^l v^l/2$$

Implementation of Case 1

modified auction: each player announces their value

the highest announced value wins; if there is a tie, flip a coin

if the low value wins, he pays his value; if the high value wins he pays

$$\frac{p^h}{q^h} = \frac{v^h/2 + \pi^l v^l/2}{\pi^l + \pi^h/2}$$

under these rules

probability that high type wins is $q^h = \pi^l + \pi^h/2$

probability that low type wins is $q^l = \pi^l/2$

just as in the optimal mechanism;

this means the expected payments are the same too

Case 2: $v^l < \pi^h v^h$

$U = q^l(v^l - \pi^h v^h) + \pi^h q^h v^h$

(1) $0 \le q^i \le \pi^{-i} + \pi^i/2$

(2) $\pi^l q^l + \pi^h q^h \le 1/2$

Make q^h large as possible, q^l as small as possible

$q^h = \pi^l + \pi^h/2$

$q^l = 0$

expected payments

$p^l = q^l v^l$, $p^h = (q^h - q^l)v^h + q^l v^l$

$p^l = 0$

$p^h = (\pi^l + \pi^h / 2)v^h$

Implementation of Case 2

set a fixed price equal to the highest valuation

$$v^h = \frac{p^h}{q^h} = \frac{(\pi^l + \pi^h/2)v^h}{\pi^l + \pi^h/2}$$

Information Aggregation in Auctions

(based on Phil Reny's slides)

(Wilson, *Restud* (1977), Milgrom, *Econometrica* (1979, 1981))

♦ n bidders, single indivisible good, 2^{nd}-price auction

♦ state of the commodity, $\omega \sim g(\omega)$, drawn from [0,1]

♦ signals, $x \sim f(x|\omega)$, drawn indep. from [0,1], given ω

♦ unit value, $v(x,\omega)$, nondecreasing (strict in x or ω)

♦ $f(x|\omega)$ satisfies strict MLRP:

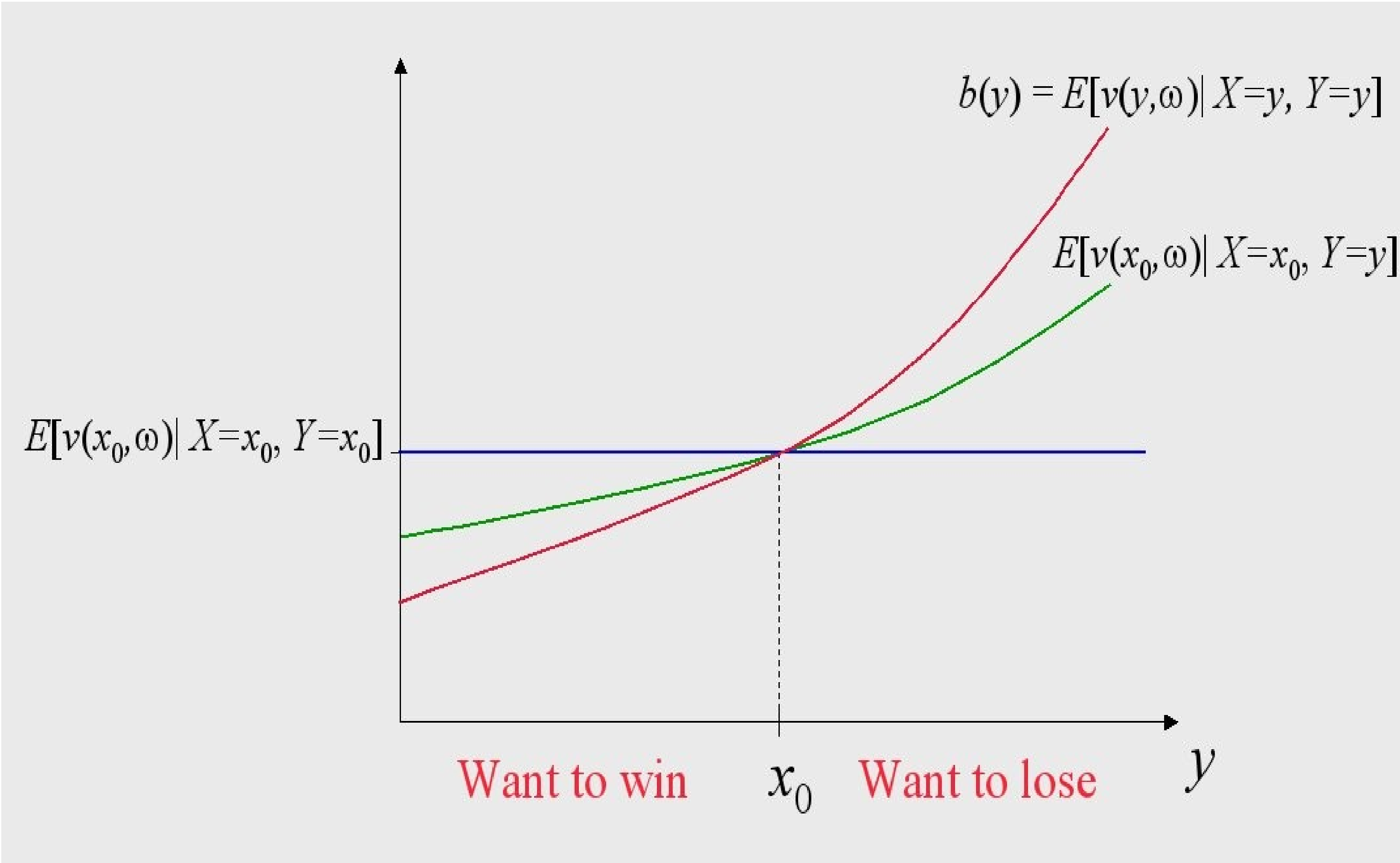
$$x > y \Rightarrow \frac{f(x \mid \omega)}{f(y \mid \omega)} \text{ strictly } \uparrow \text{ in } \omega$$

Finding the Equilibrium

- Equilibrium: $b(x) = E[v(x,\omega) | X=x, Y=x]$

    (X is owner's signal, Y is highest signal of others)

- Claim: $b(x) = E[v(x,\omega) | X=x, Y=x]$ is an equilibrium.
- Suppose signal is x_0. Is optimal bid $E[v(x_0,\omega) | X=x_0, Y=x_0]$?



$b(y) = E[v(y,\omega)|\, X=y,\, Y=y]$
$E[v(x_0,\omega)|\, X=x_0,\, Y=y]$
$E[v(x_0,\omega)|\, X=x_0,\, Y=x_0]$
Want to win
x_0
Want to lose
y

Assessing the Equilibrium

♦ Equilibrium: $b(x) = E[v(x,\omega) | X=x, Y=x]$

(X is owner's signal, Y is highest signal of others)

♦ outcome efficient for all n

♦ Equilibrium Price: $P = E[v(z,\omega) | X=z, Y=z]$,

where z is the 2nd-highest signal.

- if ω is U[0,1] and x is U[0,ω], then P->$v(\omega,\omega)$

the competitive limit, and information is aggregated.

(fails if conditional density is continuous and positive.)

Principal-Agent Problem

A risk neutral principal

A risk averse agent with utility $u(c)$, where $u(0) = 0$, $u(v) = 1$

Agent may take one of two actions $e = 0,1$ (effort level)

Total utility of agent is $u(w) - e$ where w is payment from principal

Two possible output levels $0, y$ accrue to the principal

If agent takes effort $e = 0$ then probability of y output is $\pi_0 > 0$; if agent takes effort $e = 1$ then probability is $1 > \pi_1 > \pi_0$

Assume that $\pi_1 y - 1 > \pi_0 y$ so that it is efficient for the agent to make an effort

Agent's reservation utility is 0

With complete observability

Maximize principal's utility

Pay the agent a fixed fee of v if he provides effort, nothing if he does not. So agent is indifferent gets $u(v) - 1 = 0$ if effort, $u(0) = 0$ if no effort. So he is willing to provide effort, but not if he is paid less

With incomplete observability

Principal only observes output, pays w_y, w_0

Incentive constraint for agent:

$$\pi_1 u(w_y) + (1-\pi_1)u(w_0) - 1 \geq \pi_0 u(w_y) + (1-\pi_0)u(w_0)$$

individual rationality constraint for agent:

$$\pi_1 u(w_y) + (1-\pi_1)u(w_0) - 1 \geq 0$$

Principal may pay 0, get 0, or minimize $\pi_1 w_y + (1-\pi_1)w_0$ subject to these constraints

Rewrite IC

$$\left(\pi_1 - \pi_0\right)\left[u(w_y) - u(w_0)\right] \geq 1$$

implies IR constraint must hold with equality, since otherwise could lower w_0 while maintaining IC

IR $\pi_1 u(w_y) + (1-\pi_1)u(w_0) - 1 = 0$

objective function $\pi_1 w_y + (1-\pi_1)w_0 = c$

IC $(\pi_1 - \pi_0)\left[u(w_y) - u(w_0)\right] \geq 1$

$w_y = \dfrac{c-(1-\pi_1)w_0}{\pi_1}$ [from objective function]

substitute objective into IR

$$\pi_1 u\left(\frac{c-(1-\pi_1)w_0}{\pi_1}\right) + (1-\pi_1)u(w_0) - 1 = 0$$

differentiate

$$\frac{dc}{dw_0} = -\frac{-\pi_1(1-\pi_1)u'(w_y) + (1-\pi_1)u'(w_0)}{\pi_1 u'(w_y)}$$

$$= -\frac{(1-\pi_1)\left[u'(w_0) - \pi_1 u'(w_y)\right]}{\pi_1 u'(w_y)} \leq 0$$

non-negative since $w_y \geq w_0$ implies $u'(w_0) \geq u'(w_y)$

because $dc/dw_0 \leq 0$ should increase w_0 until the IC binds

combining the IC binding with the IR

$$(\pi_1 - \pi_0)(1 - u(w_0)) = \pi_1$$

which is possible only if $u(w_0) < 0$, that is $w_0 < 0$

notice that IC implies $w_y > w_0$ so no full insurance what if constrained to $w_0 \geq 0$? ("limited liability *ex post*")

The constraint binds, so optimum has $w_0 = 0$

(IC) $(\pi_1 - \pi_0)\, u(w_y) \geq 1$

(IR) $\pi_1 u(w_y) \geq 1$ does not bind if (IC) holds

so objective is to minimize $\pi_1 w_y$ subject to IC

namely IC should bind $(\pi_1 - \pi_0)\, u(w_y) = 1$

agent earns an "informational" rent because IR does not bind

since IC binds, still have $w_y > w_0$ and no full insurance

Macro Mechanism Design: The Insurance Problem

Kehoe, Levine and Prescott [2000]

continuum of traders ex ante identical

two goods $j = 1,2$

c_j consumption of good *j*

utility is given by $\tilde{u}_1(c_1) + \tilde{u}_2(c_2)$

each household has an independent 50% chance of being in one of two states, $s = 1,2$

endowment of good 1 is state dependent

$\omega_1(2) > \omega_1(1)$

endowment of good 2 fixed at ω_2.

In the aggregate: after state is realized half of the population has high endowment half low endowment

Gains to Trade

after state is realized

low endowment types purchase good 1 and sell good 2

before state is realized

traders wish to purchase insurance against bad state

unique first best allocation

all traders consume $(\omega_1(1)+\omega_1(2))/2$ of good 1, and ω_2 of good 2.

Private Information

idiosyncratic realization private information known only to the household

first best solution is not incentive compatible

low endowment types receive payment

$(\omega_1(2) - \omega_1(1))/2$

high endowment types make payment of same amount

high endowment types misrepresent type to receive rather than make payment

Incomplete Markets

prohibit trading insurance contracts

consider only trading ex post after state realized

resulting competitive equilibrium

- equalization of marginal rates of substitution between the two goods for the two types

- low endowment type less utility than the high endowment type

Mechanism Design

purchase $x_1(1) > 0$ in exchange for $x_1(2) < 0$

no trader allowed to buy a contract that would later lead him to misrepresent his state

assume endowment may be revealed voluntarily, so low endowment may not imitate high endowment

incentive constraint for high endowment

$$\begin{aligned}&\tilde{u}_1(\omega_1(2) + x_1(2)) + \tilde{u}_2(\omega_2 + x_2(2)) \\ &\geq \tilde{u}_1(\omega_1(2) + x_1(1)) + \tilde{u}_2(\omega_2 + x_2(1))\end{aligned}$$

- Pareto improvement over incomplete market equilibrium possible since high endowment strictly satisfies this constraint at IM equilibrium

Need to monitor transactions

Lotteries and Incentive Constraints

one approach: X space of triples of net trades satisfying incentive constraint

use this as consumption set

enrich the commodity space by allowing sunspot contracts (or lotteries)

1) X may fail to be convex

2) incentive constraints can be weakened - they need only hold on average

$$E\,|_2\ \tilde{u}_1(\omega_1(2) + x_1(2)) + \tilde{u}_2(\omega_2 + x_2(2))$$
$$\geq E\,|_1\ \tilde{u}_1(\omega_1(2) + x_1(1)) + \tilde{u}_2(\omega_2 + x_2(1))$$

Other Applications of Mechanism Design

- general equilibrium theory
- public goods
- taxation
- price discrimination

Common versus Individual Punishment

N choose $a_i \in \{0, 1\}$ effort to contribute to a public good (equals cost)

no effort, no input $y_i = 0$

effort, probability $1 - \pi$ of input

let M be the number who contribute, then contributors get

$$(M(1 - \pi)/N)V - 1$$

non-contributors get

$$(M(1 - \pi)/N)V$$

where $V(1 - \pi) > 1$

suppose also that $V(1 - \pi)/N < 1$ so no voluntary contributions

Crime and Punishment

A punishment P

common punishment: if the punishment occurs everyone is punished

(will cancel future public goods projects...)

individual punishment: each individual may be punished or not separately

Common Punishment under Certainty

$\pi = 0$

if everyone contributes no punishment

otherwise punishments

incentive compatibility

$V - 1 \geq ((N - 1)/N)V - P$

or

$P \geq 1 - V/N$

expected cost of the punishment is zero

Common Punishment under Uncertainty

Probability someone doesn't contribute is $1-\pi^N$

incentive compatibility

$$V(1-\pi)-1-(1-\pi^{N-1})P \geq ((N-1)/N)(1-\pi)V-P$$

or

$$P \geq (1-V(1-\pi)/N)/\pi^{N-1}$$

expected cost of punishment

$$(1-\pi^N)(1-V(1-\pi)/N)/\pi^N$$

goes to infinity as $N \to \infty$

Theorem (Fudenberg, Levine and Pesendorfer)

People are always trying to figure a way around this

(perpetual motion machine of economics)

Suppose that P is bounded above

for any mechanism public good production goes to zero as $N \to \infty$

Individual Punishment

Punish if $y_i = 0$

incentive constraint

$$V(1-\pi) - 1 - \pi P \geq ((N-1)/N)V(1-\pi) - P$$

or

$$P \geq (1 - V(1-\pi)/N)/(1-\pi)$$

with expected cost of punishment

$$\pi(1 - V(1-\pi)/N)/(1-\pi)$$

less than $V(1-\pi) - 1$ then produce the public good