Decision Theory

Lotteries and Expected Utility

Luce, D. and H. Raiffa [1957]: *Games and Decisions,* John Wiley chapter 2.5

there are r prizes $1,\ldots,r$

a lottery L consists of a finite vector $(p_1,\ldots,p_r)$ where p_i is the "probability" of winning prize i

properties of "probabilities" $p_i \geq 0, \sum_{i=1}^{r} p_i = 1$

Definition: the lottery L_i has $p_i = 1$

Preferences $\preceq$ are defined over the set of lotteries

order the lotteries so that $L_i \succeq L_{i+1}$, that is higher numbered prizes are worse

Usual preference assumptions:

1) transitivity

2) continuity: for each L_i there exists a lottery $\tilde{L}_i$ such that $p_j = 0$ for $j = 2, \ldots, r-1$ and $L_i \sim \tilde{L}_i$

(in words: we can find probabilities of the best and worst prize that are indifferent to any lottery)

Definition: u_i is such that $\tilde{L}_i = (u_i, 0, \ldots, 0, (1-u_i))$

Assumptions relating to probability:

a compound lottery is a lottery in which the prizes are lotteries

we can write a compound lottery $(q^1, L^1, q^2, L^2, \ldots, q^k, L^k)$ where

q^i is the probability of lottery L^i (not to be confused with L_i)

1) reduction of compound lotteries

preferences are extended from simple lotteries to lotteries over lotteries by the usual laws of probability

example: $L^1 = (p_1^1, p_2^1, \ldots, p_r^1)$, $L^2 = (p_1^2, p_2^2, \ldots, p_r^2)$

$(q_1, L^1, q_2, L^2) \sim (q^1 p_1^1 + q^2 p_1^2, q^1 p_2^1 + q^2 p_2^2, \ldots, q^1 p_r^1 + q^2 p_r^2)$

2) substitutability (independence of irrelevant alternatives)

for any lottery L the compound lottery that replaces L_i with $\tilde{L}_i$ is indifferent to L

$(p_1, p_2, \ldots, p_r) \sim (p_1, L_1, p_2, L_2, \ldots, p_i, \tilde{L}_i, \ldots, p_r, L_r)$

3) monotonicity

$(p, 0, \ldots, 0, (1-p)) \succeq (p', 0, \ldots, 0, (1-p'))$ if and only if $p \succeq p'$

Expected utility theory:

Start with a lottery $L = (p_1, \ldots, p_r)$

Using transitivity and continuity L is indifferent to the compound lottery $(p_1 \tilde{L}_1, \ldots, p_r \tilde{L}_r)$

Notice that the lotteries $\tilde{L}_i$ involve only the highest and lowest prizes

Now apply reduction of compound lotteries: this is equivalent to the lottery

$L \sim (u, 0, \ldots, 0, (1-u))$ where $u = \sum_{i=1}^{r} p_i u_i$

This says that we may compare lotteries by comparing their “expected utility” and by monotonicity, higher utility is better

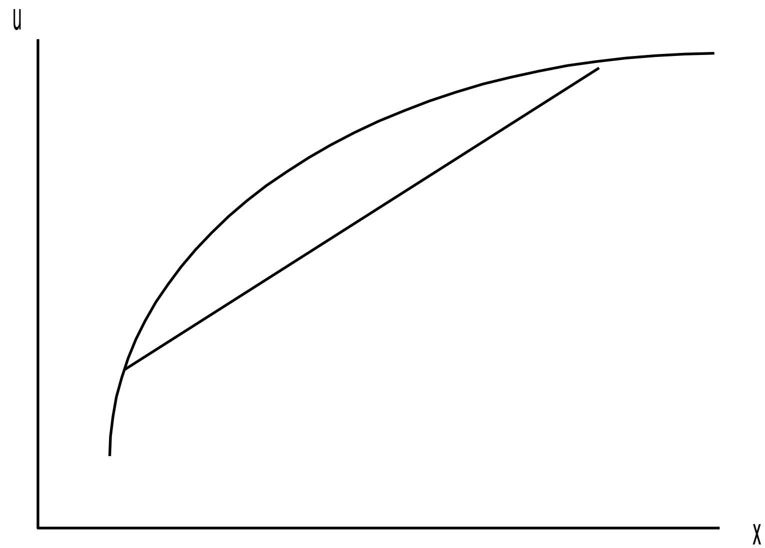
Risk Aversion

Jensen's inequality

u is a concave function if and only if $u(Ex) \geq Eu(x)$

that is: you prefer the certainty equivalent

so concavity = risk aversion


u
x

Risk premium

y a random income with $Ey = 0, Ey^2 = 1$

$u(x - p) = Eu(x + \sigma y)$

Taylor series expansion:

$$u(x) - pu'(x) = E\left[\, u(x) + \sigma u'(x)y + (1/2)\sigma^2 u''(x)y^2 \,\right]$$
$$= u(x) + (1/2)\sigma^2 u''(x)$$

so $p = -\dfrac{u''(x)}{u'(x)}\dfrac{\sigma^2}{2}$

we can also consider the relative risk premium

$u(x - \rho x) = Eu(x + \sigma y x)$

$$\rho = -\frac{u''(x)x}{u'(x)}\frac{\sigma^2}{2}$$

Measures of Risk Aversion

Absolute risk aversion

The coefficient of absolute risk aversion is $-\dfrac{u''(x)}{u'(x)}$

Relative risk aversion

The coefficient of relative risk aversion is $-\dfrac{u''(x)x}{u'(x)}$

Changes in Risk Aversion with Wealth

We ordinarily think of absolute risk aversion as declining with wealth (this is a condition on the third derivative of u).

Constant relative risk aversion

$u(x) = \frac{x^{1-\rho}}{1-\rho}$ also known as "constant elasticity of substitution" or CES

$\rho \geq 0$

$-\frac{u''(x)x}{u'(x)} = \frac{\rho x^{-\rho-1}x}{x^{-\rho}} = \rho$

$\rho = 0$ linear, risk neutral

$\rho = 1$ $u(x) = \log(x)$

useful for empirical work and growth theory

note that constant relative risk aversion implies declining absolute risk aversion