Notes on Asset Pricing and the Equity Premium Puzzle[*]
Course Econ 201B

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1 Introduction

As you are probably aware from course 202A (or any other course featuring RBC models), a large part of contemporary Macroeconomics is concerned with formulating dynamic general equilibrium models that do well when confronted to data. Most of the time, data comes from the post-war period in the US.

Usually, this data is about quantities, not prices. I guess this is because it is quantities that go into utility functions and make people happy. In general, the RBC literature measures its success by comparing the autocorrelations, covariances and volatilities of consumption, investment, output, labor input, etc. generated by the model to these same statistics in the real world data.[1]

One problem with real quantities is that they are very difficult to measure. Consumption, investment, hours worked and output have to be estimated by using incomplete data or surveys. These series are usually not available in frequencies lower than a quarter and have to be revised frequently. On the other hand, price data is fairly easy to measure and available at higher frequencies. Particularly, prices for assets traded in a stock or bond market can be measured very easily and are available even at daily frequencies or sometimes even at intra-day frequencies. It makes sense then to subject our models to the test of predicting price data correctly.

[*]Note to the typical reader: You probably couldn't care less about asset pricing. After all, you are enrolled in a Game Theory course. But even if you are not particularly interested in asset pricing or finance right now, don't throw away these notes just yet. It is highly probable that you will encounter asset pricing at some time during your PhD at UCLA. And it is even more likely that some professor will simply assume that you have seen it before. In that case it may be useful to read these notes again.

[1]One exception is the international RBC literature where one of the targeted variables is the exchange rate, which is a price.

The RBC literature restricts utility functions to lie in the CES (power utility) family. The reason for doing this is that it is the only class of preferences consistent with a balanced growth path. What happens if we use CES utility functions to predict equity and bond prices? The answer was given by Mehra and Prescott (1985) who were the first ones to state the Equity Premium Puzzle.

The Equity Premium Puzzle states that the excess return of the stock market over a risk free rate is incompatible with a representative agent model with a power utility function for "reasonable" values of risk aversion. By reasonable values of risk aversion low values are meant, possibly a value lower than one. Since 1985, there has been some progress in fitting models to financial data. Kocherlakota (1996) and Campbell (2000) provide very good surveys of these attempts.

2 How does the market react to good news?[2]

This section constructs a model of an endowment economy and argues that the risk aversion parameter should be smaller than 1 if we expect the stock market to rise with good news. In a CES function the risk aversion parameter is tied together with the intertemporal elasticity of substitution, which measures how willing consumers are to substitute consumption across time. It is the interpretation of this parameter as the intertemporal elasticity that drives this result, not the interpretation of risk aversion.

2.1 A 2 period example

Consider a deterministic two period example. Call the discount factor $\delta < 1$. In a CES function the same parameter can be interpreted as the risk-aversion parameter and the inverse of the intertemporal elasticity of substitution (IES). In this example there is no uncertainty, and therefore, it is reasonable to interpret it as the IES.

The model has a representative agent who maximizes a time separable utility

function

$$\max_{c_1,c_2} u(c_1) + \delta u(c_2) \tag{1}$$

subject to a budget constraint

$$c_1 + pc_2 \leq I \equiv \omega_1 + p\omega_2 \tag{2}$$

[2] This entire section is based on David Levine's lecture notes on Decision Theory.

where p is the price of period 2 consumption in terms of period 1 consumption, I is lifetime income, which is made out of endowments ω_1 in period 1 and ω_2 in period 2.

Assuming an interior solution, the first order condition for the consumer is:

$$\frac{\delta u'(c_2)}{u'(c_1)} = p \tag{3}$$

Consider a CES function

$$U = c_1^{1-\gamma} + \delta c_2^{1-\gamma} \tag{4}$$

Then, the first order condition becomes

$$\delta \left(\frac{c_2}{c_1}\right)^{-\gamma} = p \tag{5}$$

Solve for c_1 in terms of c_2 and p to get

$$c_1 = \left(\frac{p}{\delta}\right)^{\frac{1}{\gamma}} c_2 \tag{6}$$

Plug this into the budget constraint to get

$$\left(\frac{p}{\delta}\right)^{\frac{1}{\gamma}} c_2 + pc_2 = I \tag{7}$$

Thus, solving for consumption, we have

$$c_2 = \frac{I}{p + \left(\frac{p}{\delta}\right)^{\frac{1}{\gamma}}} \tag{8}$$

and

$$c_1 = \frac{\left(\frac{p}{\delta}\right)^{\frac{1}{\gamma}} I}{p + \left(\frac{p}{\delta}\right)^{\frac{1}{\gamma}}} \tag{9}$$

Notice that for $\gamma = 1$ (the Cobb-Douglas case) we have

$$c_2 = \frac{\delta}{1+\delta}\frac{I}{p} \tag{10}$$

and

$$c_1 = \frac{1}{1+\delta} I \tag{11}$$

Now fix $\omega_1 = 1$ and consider $\omega_2 = g \ \left(= \frac{\omega_2}{\omega_1}\right)$. At what prices is autarky, i.e. $c_1 = \omega_1 = 1$ and $c_2 = \omega_2 = g$ an equilibrium?

$$p = \frac{\delta u'(\omega_2)}{u'(\omega_1)} = \delta \left(\frac{\omega_2}{\omega_1}\right)^{-\gamma} = \frac{\delta}{g^{\gamma}} \tag{12}$$

Check that $c_1 = \omega_1 = 1$ and $c_2 = \omega_2 = g$.

$$c_2 = \frac{1 + \frac{\delta}{g^{\gamma}} g}{\frac{\delta}{g^{\gamma}} + \left(\frac{\frac{\delta}{g^{\gamma}}}{\delta}\right)^{\frac{1}{\gamma}}} = \frac{1 + \frac{\delta}{g^{\gamma}} g}{\frac{\delta}{g^{\gamma}} + \frac{1}{g}} = \frac{1 + \delta g^{1-\gamma}}{1 + \delta g^{1-\gamma}} g = g \tag{13}$$

and

$$c_1 = \left(\frac{\frac{\delta}{g^{\gamma}}}{\delta}\right)^{\frac{1}{\gamma}} g = 1 \tag{14}$$

What is the value of the stock market?[3]

$$p\omega_2 = pc_2 = \left(\delta g^{-\gamma}\right) g = \delta g^{1-\gamma} \tag{15}$$

and

$$\frac{\partial p\omega_2}{\partial g} = \delta (1-\gamma) g^{-\gamma} < 0 \text{ if } \gamma > 1 \tag{16}$$

Therefore, if $\gamma > 1$ then good news about tomorrow's endowment imply that the value of the stock market decreases, which is counterfactual. The reason is simple. An increase of the endowment in period 2 makes consumption in period 2 more abundant. In autarky, its relative price has to fall so that the representative agent demands more of that good. The price will need to fall more the less substitutable the goods are. For high substitutability a small change in the price is already enough to induce the agent to consume more in period 2. The limiting case is when we have $\gamma = 1$ (Cobb Douglas). In the Cobb-Douglas case the expenditure on each good is a constant fraction of income and income does not change since the increase in ω_2 is exactly offset by the decrease in p.

2.1.1 Intertemporal elasticity of substitution

Another way of analyzing this is through the IES.

We can write the derivative we are interested in as follows

[3] Income is $\omega_1 + p\omega_2 = 1 + p\omega_2$. When we think of the value of the stock market we only consider period 2. Period 1 is irrelevant since its contribution to lifetime income is fixed at 1.

$$
\begin{aligned}
\frac{\partial p\omega_2}{\partial g} &= \frac{\partial p}{\partial g}\omega_2 + p\frac{\partial \omega_2}{\partial g} = \frac{\partial p}{\partial g}g + p\frac{\partial g}{\partial g} = \frac{\partial p}{\partial g}g + p = \\
&= p\left(1 + \frac{\partial p}{\partial g}\frac{g}{p}\right) = p\left(1 + \frac{\partial p}{\partial\left(\frac{c_2}{c_1}\right)}\frac{\left(\frac{c_2}{c_1}\right)}{p}\right) = p\left(1 - \frac{1}{IES}\right)
\end{aligned} \tag{17}
$$

The IES can be calculated from the price equation,

$$
p = \frac{\delta}{\left(\frac{c_2}{c_1}\right)^{\gamma}} \Leftrightarrow \ln p = \ln \delta - \gamma \ln\left(\frac{c_2}{c_1}\right) \tag{18}
$$

Therefore,

$$
IES = -\frac{\partial \ln\left(\frac{c_2}{c_1}\right)}{\partial \ln(p)} = \frac{1}{\gamma} \tag{19}
$$

Then, the equation we get is

$$
\frac{\partial p\omega_2}{\partial g} = p(1-\gamma) \tag{20}
$$

The Cobb-Douglas function has $IES = 1$. Consequently, the effect on the stock market is zero. For values of $\gamma \neq 1$ we get either positive or negative responses.

More intuition on the 2 good example is offered in Appendix 0.

2.2 Infinite Periods

The 2 period example can obviously be generalized to infinite periods.

In that case the price for period t consumption in terms of period 1 would be

$$
p_t = \frac{\delta^{t-1}u'(c_{t+1})}{u'(c_1)} \tag{21}
$$

Therefore, a claim to all future endowments would be calculated as

$$
\sum_{t=2}^{\infty} p_t\omega_t = \sum_{t=2}^{\infty}\frac{\delta^{t-1}u'(c_{t+1})}{u'(c_1)}\omega_t \tag{22}
$$

Considering the case where the endowment starts at 1 and grows at a constant rate g we have $\omega_t = g^{t-1}$. With a CES function and at autarky

$$
p_t = \frac{\delta^{t-1}\omega_{t+1}^{-\gamma}}{\omega_1^{-\gamma}} = \frac{\delta^{t-1}g^{-\gamma(t-1)}}{1} = \left(\frac{\delta}{g^{\gamma}}\right)^{t-1} \tag{23}
$$

Hence,

$$\sum_{t=2}^{\infty} p_t \omega_t = \sum_{t=2}^{\infty} \left(\frac{\delta}{g^{\gamma}}\right)^{t-1} g^{t-1} = \sum_{t=1}^{\infty} \left(\delta g^{1-\gamma}\right)^{t} \tag{24}$$

If $\delta g^{1-\gamma} < 1$

$$\sum_{t=2}^{\infty} p_t \omega_t = \frac{\delta g^{1-\gamma}}{1 - \delta g^{1-\gamma}} \tag{25}$$

Now consider how this changes with good news

$$\frac{\partial}{\partial g} \left[\sum_{t=2}^{\infty} p_t \omega_t\right] = \frac{(1-\gamma)\, \delta g^{-\gamma}}{\left(1 - \delta g^{1-\gamma}\right)^2} \tag{26}$$

Therefore, we obtain that the value of the stock market changes according to the sign of $1-\gamma$, just as before.

3 Equity Premium Puzzle. The model

In this section the interest is on the CES parameter as a measure of risk averision. Thus, uncertainty is introduced.

3.1 The state space

Time is discrete. Let S_t be the state space at date t with generic element s_t. A history is denoted by $s^t = (s_0, s_1, ..., s_t)$ for $t \geq 0$. Histories are partially ordered. We say that $\tilde{s}^{t+1} \geq \tilde{s}^t$ if all the elements of both histories up to the date t element, $\tilde{s}_t$, coincide. The model is populated by a representative agent who faces uncertainty about the occurence of histories. His assessment of the probability of history s^t is $\pi(s^t)$. We can also define the conditional probability $\pi\left(s^{t+1}|s^t\right) = \frac{\pi(s^{t+1})}{\pi(s^t)}$ where actually

$$\pi\left(s^{t+1}|s^t\right) = \begin{cases} \pi\left(s_{t+1}|s^t\right) \geq 0 & \text{if } s^{t+1} \geq s^t \\ 0 & \text{otherwise} \end{cases}$$

Sometimes histories are referred to as "nodes". This comes from the fact that the evolution of states is sometimes represented by the use of a tree similar to the tree of an extensive form game in game theory. If you take any node in that tree, there is a single history s^t which leads to that node.

3.2 Asset structure

The representative agent has access to a complete set of Arrow securities in zero net supply and to a risky asset (a Lucas tree) in positive supply. The Arrow security denoted by $a\left(s^{t+1}\right)$ has a price of $q\left(s^{t+1}\right)$ at date t and delivers one unit of the consumption good at date $t+1$ if history s^{t+1} is realized, and zero in all other histories.

The risky asset in the economy is indexed by m, i.e. holdings of this asset at the beginning of date t in history s^t will be denoted as $a^m\left(s^t\right)$. This asset has a $t+1$ dividend $d(s^{t+1})$ which is unknown at date t. The price at date t for this asset will be denoted as $p\left(s^t\right)$. We will assume that the set of assets completely spans the state space and rule out default.

3.3 Goods and Preferences

There is a single consumption good in the economy. (Actually there are ∞ goods, one in each period in each history).

A representative consumer orders streams of consumption $\{c\left(s^t\right)\}$ using the following time separable utility function.

$$U=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t u\left(c\left(s^t\right)\right)\pi\left(s^t\right) \tag{27}$$

The instantaneous utility function is of the constant relative risk aversion (CRRA) type

$$u\left(x\right)=\begin{cases}\frac{x^{1-\gamma}}{1-\gamma} & \text{if } \gamma\neq 1\\ \ln\left(x\right) & \text{otherwise}\end{cases} \tag{28}$$

with $\gamma\geq 0$.[4]

At each history s^t the agent is endowed with $y\left(s^t\right)$ of the good.

Consequently, the budget constraint for the agent at node s^t is

$$c\left(s^t\right)+\sum_{s^{t+1}\geq s^t}q\left(s^{t+1}\right)a\left(s^{t+1}\right)+p\left(s^t\right)a^m\left(s^{t+1}\right)\leq y\left(s^t\right)+a\left(s^t\right)+\left[p\left(s^t\right)+d\left(s^t\right)\right]a^m\left(s^t\right) \tag{29}$$

with the restriction that $a\left(s^{t+1}\right)=a\left(\tilde{s}^{t+1}\right)$ if $s^{t+1}\geq s^t$ and $\tilde{s}^{t+1}\geq s^t$. This last restriction makes sense since $a\left(s^{t+1}\right)$ is a choice variable at date t, when the realization of s_{t+1} is still unknown. Therefore, the agent has to carry the same amount of the asset into each future state that might come after history s^t.

[4] In class, Daisuke kindly provided us with the proof that $\lim_{\gamma\to 1}\frac{x^{1-\gamma}}{1-\gamma}=\log x$. By L'Hopital's rule $\lim_{\gamma\to 1}\frac{x^{1-\gamma}}{1-\gamma}=\lim_{\gamma\to 1}\frac{-x^{1-\gamma}\log x}{-1}=x^{1-1}\log x=\log x$.

3.4 Equilibrium

An equilibrium for this economy is a sequence of allocations $\left\{c\left(s^t\right), \left\{a\left(s^{t+1}\right)\right\}, a^m\left(s^t\right)\right\}_{s^t}$, asset prices $\left\{\left\{q\left(s^{t+1}\right)\right\}, p\left(s^t\right)\right\}_{s^t}$ for an income process $\left\{y_t\left(s^t\right)\right\}_{s^t}$ such that (i) the agent chooses $\left\{c\left(s^t\right), \left\{a\left(s^{t+1}\right)\right\}, a^m\left(s^t\right)\right\}$ to maximize (27) subject to (29) and given $a^m\left(s^0\right)$, and (ii) the markets for assets clear.

3.5 Deriving the Asset Pricing Equation

The asset pricing equation is an alternative name for the Euler equation. Setting up the Lagrangian for the consumer,

$$\begin{aligned} L &= \sum_{t=0}^{\infty}\sum_{s^t}\{\left[\beta^t u\left(c\left(s^t\right)\right)\pi\left(s^t\right)\right] + \lambda\left(s^t\right)\left[y\left(s^t\right) + a\left(s^t\right) + \left[p\left(s^t\right) + d\left(s^t\right)\right]a^m\left(s^t\right)\right. \\ &\quad \left. -c\left(s^t\right) - \sum_{s^{t+1}\geq s^t} q\left(s^{t+1}\right)a\left(s^{t+1}\right) - p\left(s^t\right)a^m\left(s^{t+1}\right)\right]\} \end{aligned} \tag{30}$$

By taking partial derivatives $\frac{\partial L}{\partial c(s^t)} = 0$ and $\frac{\partial L}{\partial c(s^{t+1})} = 0$ we obtain

$$\beta^t u'\left(c\left(s^t\right)\right)\pi\left(s^t\right) = \lambda\left(s^t\right) \tag{31}$$

and

$$\beta^{t+1} u'\left(c\left(s^{t+1}\right)\right)\pi\left(s^{t+1}\right) = \lambda\left(s^{t+1}\right) \tag{32}$$

Calculating $\frac{\partial L}{a^m(s^{t+1})} = 0$

$$\lambda\left(s^t\right)p\left(s^t\right) = \sum_{s^{t+1}\geq s^t}\lambda\left(s^{t+1}\right)\left[p\left(s^{t+1}\right) + d\left(s^{t+1}\right)\right] \tag{33}$$

Calculating $\frac{\partial L}{a(s^{t+1})} = 0$

$$\lambda\left(s^t\right)q\left(s^{t+1}\right) = \lambda\left(s^{t+1}\right) \tag{34}$$

Plugging (31) and (32) into (33) we obtain

$$\sum_{s^{t+1}\geq s^t}\beta u'\left(c\left(s^{t+1}\right)\right)\left[p\left(s^{t+1}\right) + d\left(s^{t+1}\right)\right]\pi\left(s^{t+1}|s^t\right) = u'\left(c\left(s^t\right)\right)p\left(s^t\right) \tag{35}$$

which can be rearranged as

$$\sum_{s^{t+1}\geq s^t}\left[\frac{\beta u'\left(c\left(s^{t+1}\right)\right)}{u'\left(c\left(s^t\right)\right)}\right]\left[\frac{p\left(s^{t+1}\right)+d\left(s^{t+1}\right)}{p\left(s^t\right)}\right]\pi\left(s^{t+1}|s^t\right)=1 \tag{36}$$

Define the stochastic discount factor (SDF) in the usual way

$$m\left(s^t,s_{t+1}\right)=\frac{\beta u'\left(c\left(s^{t+1}\right)\right)}{u'\left(c\left(s^t\right)\right)} \tag{37}$$

and express (36) in terms of expectations to get

$$E\left[m\left(s^t,s_{t+1}\right)\frac{p\left(s^{t+1}\right)+d\left(s^{t+1}\right)}{p\left(s^t\right)}|s^t\right]=1 \tag{38}$$

Now realize that the return on the risky asset is given by

$$R^m\left(s^t,s_{t+1}\right)\equiv\frac{p\left(s^{t+1}\right)+d\left(s^{t+1}\right)}{p\left(s^t\right)} \tag{39}$$

Finally, we can write the Pricing Equation for the risky asset as

$$E\left[m\left(s^t,s_{t+1}\right)R^m\left(s^t,s_{t+1}\right)|s^t\right]=1 \tag{40}$$

3.6 The risk-free rate

Plugging (31) and (32) into (34) we obtain

$$q\left(s^{t+1}\right)=\frac{\beta u'\left(c\left(s^{t+1}\right)\right)}{u'\left(c\left(s^t\right)\right)}\pi\left(s^{t+1}|s^t\right) \tag{41}$$

Now sum (41) over all s^{t+1} to get

$$\sum_{s^{t+1}\geq s^t}q\left(s^{t+1}\right)=\sum_{s^{t+1}\geq s^t}\frac{\beta u'\left(c\left(s^{t+1}\right)\right)}{u'\left(c\left(s^t\right)\right)}\pi\left(s^{t+1}|s^t\right) \tag{42}$$

Rewrite it in terms of expectations and using the definition of the SDF to get

$$\sum_{s^{t+1}\geq s^t}q\left(s^{t+1}\right)=E\left[m\left(s^t,s_{t+1}\right)|s^t\right] \tag{43}$$

Now we can think how to calculate the risk-free rate in terms of the prices of the Arrow securities. A risk-free bond delivers 1 unit of consumption in every possible state. We can construct such a bond by purchasing one unit of every Arrow security. The cost of purchasing all these Arrow securities is

$\sum_{s^{t+1}} q\left(s^{t+1}\right)$ and we obtain 1 for sure at date $t+1$. Therefore, the risk-free rate is

$$R^f\left(s^t, s_{t+1}\right) \equiv \frac{1}{\sum_{s^{t+1} \geq s^t} q\left(s^{t+1}\right)}$$

The risk-free rate does not really depend on s_{t+1} since $q\left(s^{t+1}\right)$ is determined at t. We can write it as $R^f\left(s^t\right)$. We can now rewrite (43) as

$$R^f\left(s^t\right) = \frac{1}{\sum_{s^{t+1} \geq s^t} q\left(s^{t+1}\right)} = \frac{1}{E\left[m\left(s^t, s_{t+1}\right)|s^t\right]} \tag{44}$$

or alternatively,

$$E\left[m\left(s^t, s_{t+1}\right)|s^t\right] R^f\left(s^t\right) = 1 \tag{45}$$

Notice the similarity to the Pricing Equation for the risky asset. Sometimes the moment conditions for both assets and are substracted to express the excess return as

$$E\left[m\left(s^t, s_{t+1}\right)\left(R^m\left(s^t, s_{t+1}\right) - R^f\left(s^t\right)\right)|s^t\right] = 0 \tag{46}$$

4 Taking the model to data

4.1 Choosing the utility function

If we want to be consistent with a Balanced Growth Path we are forced to choose CES function. Then, the moment conditions become

$$E\left[\left(\frac{c\left(s^{t+1}\right)}{c\left(s^t\right)}\right)^{-\gamma} R^m\left(s^t, s_{t+1}\right)|s^t\right] = 1 \tag{47}$$

$$E\left[\left(\frac{c\left(s^{t+1}\right)}{c\left(s^t\right)}\right)^{-\gamma}|s^t\right] R^f\left(s^t\right) = 1 \tag{48}$$

4.2 Approaches

There are several ways to approach the problem of finding a value of γ consistent with the pricing equations.

4.2.1 Econometrics

We might use econometrics to estimate γ using the two moment conditions. We could estimate γ using method of moments, for example.

4.2.2 Calibration of a General Equilibrium Model

This method uses all the equilibrium equations, not only the Euler equation. It is more complicated and requires the use of a computer.

4.2.3 Other approaches

Another approach is making simplifying assumptions to solve for γ explicitly. In what follows we will assume log-normal returns and consumption growth as in Hansen and Singleton (1983). This approach is used in the next section.

5 The Log-normal Case

This section assumes a log-normal joint process for the market return, private consumption and the aggregate good and derives an expression for the equity premium that can be contrasted with data. This way of tackling the problem is similar to Hansen and Singleton (1983).

Consider the moment condition for the risk-free rate and the market return.

$$E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}R^j_{t+1}\right]=1 \qquad j=f,m \tag{49}$$

Suppose that $\frac{c_{t+1}}{c_t}$ and R^m_{t+1} are random variables which follow a conditional joint log-normal process

$$\begin{pmatrix}\log(\frac{c_{t+1}}{c_t})\\ \log(R^m_{t+1})\end{pmatrix}\sim N\left[\begin{pmatrix}\mu_c\\ \mu_R\end{pmatrix},\begin{pmatrix}\sigma_c^2 & \sigma_{cR}\\ \sigma_{cR} & \sigma_R^2\end{pmatrix}\right] \tag{50}$$

Define the equity premium as $EP\equiv E\left[R^m-R^f\right]$.Then we have the following proposition.

Proposition 1 *If $\frac{c_{t+1}}{c_t}$ and R^m_{t+1} are random variables which follow the conditional joint log-normal process in* (50) *and satisfy the moment condition* (49) *then the equity premium has to satisfy the following equation*

$$EP=\gamma\sigma_{cR} \tag{51}$$

Proof. In Appendix 1. ■

Solving for γ from the previous equation

$$\gamma = \frac{EP}{\sigma_{cR}} \tag{52}$$

This expression relates the risk aversion parameter γ to numbers in the data.

6 Data

This section takes equation (52) to US data. It focuses on the post-war period (1946-2002). The data was calculated in Campos (2004) from Shiller's database and the IFS statistics. The sources for the data and all calculations that were performed are explained in detail in Appendix 2.

6.1 Asset market

The data for the market return and the risk free rate is from the dataset of Shiller (2003). For this dataset the average risky and risk free rate and the variance of the risky rate are presented in the following table.

Table 1: Returns in the data

μ_R	0.06980619
μ_{Rf}	0.010107879
σ_R^2	0.024076136

With these data the equity premium adjusted for Jensen's inequality[5] is

$$EP = 0.071736379$$

6.2 Real per capita consumption

Consumption data is from the International Financial Statistics database published by the IMF. The mean, variance and covariance with the risky rate for the growth rate of real per capita consumption is shown in the following table.

Table 2: Consumption in the data

μ_c	0.019446875
σ_c^2	0.000597982
σ_{cR}	0.00183304

[5] See Appendix 2 for the calculation.

6.3 Risk aversion

Using the formula for γ we derived in (52)

$$\gamma = \frac{EP}{\sigma_{cR}} = \frac{0.071736379}{0.00183304} = 39.135 \tag{53}$$

This value for the risk aversion parameter is larger than what most economists would find reasonable.

7 Lower Bounds on the SDF

Suppose that we want to construct a model which is not at odds with financial price data. What do we need to construct such a model? Specifically, what conditions do we need on the stochastic discount factor (SDF)?

To answer this question, return to the pricing equation involving the excess return

$$E\left[m\left(R^m - R^f\right)\right] = 0 \tag{54}$$

Using the formula for the covariance[6]

$$E\left[m\right]E\left[\left(R^m - R^f\right)\right] + Cov\left[m, R^m - R^f\right] = 0 \tag{55}$$

Divide by σ_m and σ_R to get

$$\frac{E\left[m\right]}{\sigma_m}\frac{E\left[\left(R^m - R^f\right)\right]}{\sigma_R} + \frac{Cov\left[m, R^m - R^f\right]}{\sigma_m\sigma_R} = 0 \tag{56}$$

The second expression is just the correlation of the equity premium and the SDF

$$\frac{E\left[m\right]}{\sigma_m}\frac{E\left[\left(R^m - R^f\right)\right]}{\sigma_R} + \rho\left[m, R^m - R^f\right] = 0 \tag{57}$$

Rearrange to get

$$\frac{\sigma_m}{E\left[m\right]}\rho\left[m, R^m - R^f\right] = -\frac{E\left[\left(R^m - R^f\right)\right]}{\sigma_R} \tag{58}$$

Now use the fact that $-1 \leq \rho\left[m, R^m - R^f\right] \leq 1$. Therefore,

$$\frac{\sigma_m}{E\left[m\right]} \geq \frac{\left|E\left[\left(R^m - R^f\right)\right]\right|}{\sigma_R} \tag{59}$$

[6] $Cov\left(X, Y\right) = E\left[XY\right] - E\left[X\right]E\left[Y\right]$ and therefore, $E\left[XY\right] = Cov\left(X, Y\right) + E\left[X\right]E\left[Y\right]$.

The LHS is known as the market price of risk. The RHS is known as the Sharpe Ratio. This expression states that for our model to be successful to explain the data we need a volatility of the SDF of at least the Sharpe Ratio. In the data, for the market return,

$$\frac{\left|E\left[\left(R^m - R^f\right)\right]\right|}{\sigma_R} = \frac{0.071736379}{\sqrt{0.024076136}} = 0.462\,32 \tag{60}$$

So we know that for our model to be successful

$$\frac{\sigma_m}{E\left[m\right]} \geq 0.462\,32 \tag{61}$$

References

[1] Campbell, John (2000), Asset Pricing at the Millennium, The Journal of Finance, vol. LV, no. 4, 1515-1567.

[2] Campos, Rodolfo (2004), The Equity Premium Puzzle with Government Expenditure in the Utility Function, Asset Pricing Field Paper, unpublished, UCLA.

[3] Casella, George, and Roger Berger (2001), Statistical Inference, 2nd ed.

[4] Hansen, Lars Peter and Kenneth Singleton (1983), Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns, The Journal of Political Economy, vol 91, no 2, 249-265.

[5] Kocherlakota, Narayana (1996), The equity premium: It's still a puzzle, Journal of Economic Literature 23, 42-71.

[6] Mehra, Rajnish, and Edward Prescott (1985), The equity premium: A puzzle, Journal of Monetary Economics 15, 145–161.

[7] Shiller, Robert (1989), Market Volatility, MIT Press, Chapter 26.

[8] Shiller, Robert (2003), From Efficient Markets to Behavioral Finance, Journal of Economic Perspectives, vol 17, no 1, 83-104.

Appendix 0: The obscure "Substitution cancels income effect" expression in the 2 good example

Sometimes when there is no effect on something with a Cobb-Douglas utility function, people (mostly macro-economists) use the phrase "the substitution effect exactly cancels the income effect. What does this mean? It means is that consumption in a given period just depends on lifetime income and the price of consumption of that period alone. To see how these two statements are the same we can use the cross-price Slutsky Equation to divide the total effect on good 1 of a price change in good 2 into a substitution and an income effect.

The cross-price Slutsky equation can be written as

$$\frac{\partial c_1}{\partial p} = \frac{\partial c_1^c}{\partial p} - c_2 \frac{\partial c_1}{\partial I} \tag{62}$$

The LHS term is sometimes referred to as the Total Effect (TE). The first term on the RHS is the cross-price derivative of the compensated demand function and measures the Substitution Effect (SE). The last term on the RHS is the Income Effect (IE).

$$TE = SE + IE \tag{63}$$

If we calculate the Total Effect we observe that it is zero since the demand of first period consumption does not depend on p.

$$TE = \frac{\partial c_1}{\partial p} = 0 \tag{64}$$

To solve for the SE first, find the compensated demand functions. The compensated demand functions solve the following FOC

$$\frac{\delta c_1}{c_2} = p \Leftrightarrow c_1 = \frac{p c_2}{\delta} \tag{65}$$

and the equation of the utility function

$$U = c_1 c_2^{\delta} \tag{66}$$

Plugging the FOC into the utility function we get

$$U = \frac{p c_2}{\delta} c_2^{\delta} = \frac{p}{\delta} c_2^{1+\delta} \tag{67}$$

Therefore, the compensated demand for period 2 consumption is

$$c_2^c = \left(\frac{\delta U}{p}\right)^{\frac{1}{1+\delta}} \tag{68}$$

The compensated demand for period 1 consumption is

$$c_1^c = \frac{p}{\delta}\left(\frac{\delta U}{p}\right)^{\frac{1}{1+\delta}} = \left(\frac{p}{\delta}\right)^{\frac{\delta}{1+\delta}} U^{\frac{1}{1+\delta}} \tag{69}$$

Now we can calculate the Substitution Effect

$$SE = \frac{\partial c_1^c}{\partial p} = \frac{\delta}{1+\delta}\frac{1}{\delta}\left(\frac{p}{\delta}\right)^{\frac{\delta}{1+\delta}-1} U^{\frac{1}{1+\delta}} = \frac{1}{1+\delta} c_2 \tag{70}$$

To find the Income Effect, first calculate the income derivative

$$\frac{\partial c_1}{\partial I} = \frac{1}{1+\delta} \tag{71}$$

The Income Effect is

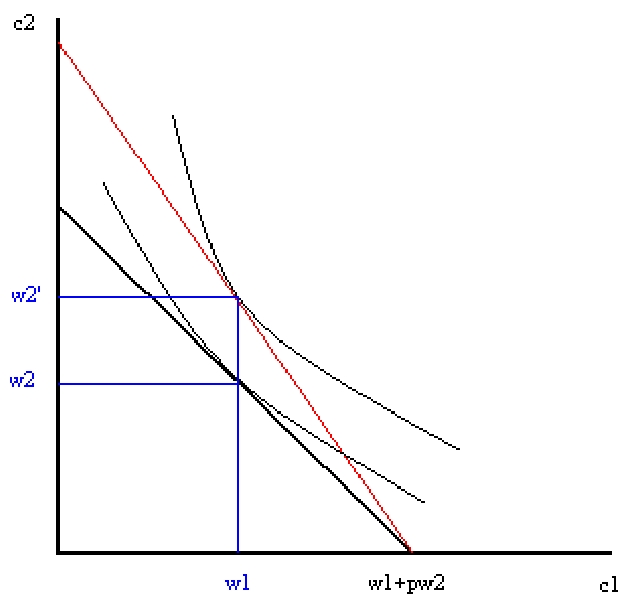
$$IE = -c_2 \frac{\partial c_1}{\partial I} = \frac{1}{1+\delta} c_2 \tag{72}$$

If we sume the SE and the IE we get the Total effect

$$TE = SE + IE = \frac{1}{1+\delta} c_2 - \frac{1}{1+\delta} c_2 = 0 \tag{73}$$

If the SE and the IE offset each other, then the Total Effect of a cross-price calculation is always zero, which means that consumption depends only on the own price.

In the following graph we can depict the experiment we just did. Disregard the measurements on the axes as they are not intended for this part. The graph shows an initial equilibrium where the lower budget line is tangent to an indifference curve. Now consider a fall of the price of c_2 for a given income. The budget line will tilt and become steeper. The new equilibrium is at such a point where consumption in the first period remains constant, hence the total effect is zero.



This same graph can also be reinterpretetd to answer what happens to lifetime income, and hence to the value of the stock market. Interpret the graph in the following way:

Start with the two endowment points and graph two indifference curves. The indifference curve that goes through (ω_1, ω_2) and the indifference curve that goes through (ω_1, ω_2'). Prices such that the optimal consumption is equal to endowments are given by the slope of the line tangent to the indifference curves at the endowments. We can measure lifetime income in period 1 terms as the intersection of a budget lines with the x-axis. In the case of Cobb-Douglas preferences the two budget lines intersect the x-axis at the same point. Therefore, lifetime income is not affected, and neither is the value of the stock market.

If we had $\gamma > 1$ then $IES < 1$ meaning that the indifference curves are more convex than Cobb-Douglas. In that case the budget line would be steeper and cross the x-axis to the left of the Cobb-Douglas case. The stock market goes down. With $\gamma < 1$ it is the other way around.

Appendix 1: Proof of Proposition 1

Using the fact that $y = \exp[\log(y)]$ equation (49) can be expressed as

$$E_t\left[\exp\left(\log\beta - \gamma\log\left(\frac{c_{t+1}}{c_t}\right) + \log R^m_{t+1}\right)\right] = 1 \tag{74}$$

Now we have an equation of the form

$$E_t\left[\exp(z)\right] = 1 \tag{75}$$

where z is normal with mean μ_z and variance σ_z^2.

$$z \sim N\left(\mu_z, \sigma_z^2\right) \tag{76}$$

The mean μ_z can be calculated as

$$\mu_z = \log\beta - \gamma\mu_c + \mu_R \tag{77}$$

and the variance σ_z^2

$$\sigma_z^2 = \gamma^2\sigma_c^2 + \sigma_R^2 - 2\gamma\sigma_{cR} \tag{78}$$

Using a well established result (see for example Casella and Berger (2001), p.109), $E_t\left[\exp(z)\right] = \exp\left(\mu_z + \frac{\sigma_z^2}{2}\right)$ whenever $z \sim N\left(\mu_z, \sigma_z^2\right)$, which yields

$$\exp\left(\mu_z + \frac{\sigma_z^2}{2}\right) = 1 \tag{79}$$

Taking logs on both sides,

$$\mu_z + \frac{\sigma_z^2}{2} = 0 \tag{80}$$

Using the calculated values for μ_z and σ_z^2 this equation becomes

$$\log\beta - \gamma\mu_c + \mu_R + \frac{\gamma^2}{2}\sigma_c^2 + \frac{\sigma_R^2}{2} - \gamma\sigma_{cR} = 0 \tag{81}$$

Notice that this equation was derived from (49) and hence also holds for the special case of the risk-free rate. In the case of the risk-free rate $\sigma_{R^f}^2 = 0$ and $\sigma_{cR^f} = 0$. Therefore, the equation for the risk-free rate is given by

$$\log\beta - \gamma\mu_c + \mu_{R^f} + \frac{\gamma^2}{2}\sigma_c^2 = 0 \tag{82}$$

The equity premium is defined as

$$EP \equiv E\left[R^m - R^f\right] \tag{83}$$

Because of the log normality assumption for R^m we have

$$EP \equiv E\left[R^m - R^f\right] = \mu_R - \mu_{R^f} + \frac{\sigma_R^2}{2} \tag{84}$$

This in turn, combining the moment ecuations derived for the risky and riskless rate, means that the equity premium can be calculated as

$$EP = \mu_R - \mu_{R^f} + \frac{\sigma_R^2}{2} = \gamma\sigma_{cR} \tag{85}$$

which is the equation in the proposition.

Appendix 2: Data sources and calculations

All data are annual and for the time frame 1946-2002. This is the same dataset as the one used in Campos (2004).

1. Returns

The data for stock prices and the riskless rate was taken from the dataset of Shiller (2003). This dataset is an update of data shown in Chapter 26 of Shiller (1989). This dataset covers the time period 1871-2003. For market return it uses the S&P 500 index and calculates the real return for that series using the CPI index. The risk-free rate is the 6-month commercial paper rate of the Federal Reserve board. After 1997, when this series was discontinued, the dataset uses the 6-month Certificate of Deposit rate, secondary market. It is converted to the real risk-free rate through the use of the CPI index.

Using the log of real market return series (Column S in Shiller's excel file), I calculated μ_R as the mean log return for the for the period 1946-2002. The variance σ_R^2 was calculated as the variance of the log return for that same period.

I calculated the mean risk-free return μ_{R^f} as the mean log real return (the exact formula is the mean over the period 1946-2002 of $\ln\left(1 + \frac{x}{100}\right)$ where x is an element of column G of Shiller's dataset).

Given the log-normality assumption the equity premium was calculated as

$$EP = \mu_R - \mu_{R^f} + \frac{\sigma_R^2}{2} \tag{86}$$

2. Private consumption

Nominal private consumption was taken from the International Financial Statistics (IFS) dataset published by the International Monetary Fund (IMF). This series is called Household Consumption Expenditure and is coded 11196F.CZF. It is available annually starting in 1948. This series was transformed into real consumption by using the CPI in Shiller's database and into real per-capita consumption by dividing by population (Series 11199Z..ZF from the IFS).

I calculated μ_c as the mean of $\ln\left(\frac{c_{t+1}}{c_t}\right)$ for the period 1948-2002, where c_t stands for real per-capita consumption. σ_c^2 was calculated as the variance of $\ln\left(\frac{c_{t+1}}{c_t}\right)$ and σ_{cR}^2 as the covariance of $\ln\left(\frac{c_{t+1}}{c_t}\right)$ and the log of the real market return over the period 1948-2002.