Notes on the computation of Nash Equilibria
Course Econ 201B

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1 Nash Equilibrium

A Nash equilibrium is a strategy profile such that each player is using a strategy that is a best response to the strategies of the other players.

A mixed strategy Nash equilibrium is a Nash equilibrium in which players are using randomized strategies, i.e. choosing among pure strategies according to probabilities.

Example 1: Prisoner's Dilemma

	L	R
T	5,5	0,7
B	7,0	1,1

1.1 Necessary and sufficient conditions for a mixed Nash equilibrium

For each player

1. All pure strategies played with positive probability in the mixed strategy (in the support) yield the same payoff
2. There is no pure strategy that yields a higher payoff than any pure strategy that is played with positive probability.

Reason for condition 1: suppose that there exists a Nash equilibrium in which two strategies that are played with positive probability have different payoffs. Then it is worthwile for the player to increase the weight given to the strategy with the higher payoff since this will increase expected utility. This means that the original mixed strategy was not a best response and hence not part of a Nash equilibrium, which is a contradiction. Therefore, it must be that in a Nash equilibrium all strategies with positive probability yield the same payoff.

Reason for condition 2: If there is a pure strategy that yields a higher payoff than any strategy in the support then it yields a higher payoff than the mixed strategy as a whole (since all strategies in the support have the same payoff). This means that the player wants to deviate and that the mixed strategy is not a best response, and hence not a Nash equilibrium. Therefore, for the mixed strategy to be part of an equilibrium Condition 2 must be fulfilled.

Example 2: Penalties in soccer (matching pennies)

	L	R
L	-1,1	1,-1
R	1,-1	-1,1

Example 3: Penalties again. Shooter is better when shooting to the left

	L	R
L	-1,1	$\frac{1}{2},-\frac{1}{2}$
R	1,-1	-1,1

2 The General Recipe for finding Nash Equilibria

1. Construct the Normal form of the game
2. Proceed iteratively to eliminate all strictly dominated pure strategies
3. Find the best responses to the opponents pure strategies for each player and check if there is a Nash equilibrium in pure strategies
4. Find all mixed strategy equilibria by listing all possible combinations for the support of mixed strategies for each player and checking the necessary and sufficient conditions for a Nash equilibrium in mixed strategies.

Example 4: Iterative elimination yields a unique Nash equilibrium

	F	G	H	I	J
A	4,4	1,6	0,0	0,0	0,0
B	6,1	2,2	3,0	0,0	0,0
C	0,0	0,1	0,0	9,0	0,0
D	0,0	0,1	0,0	0,0	0,0
E	-4,0	-3,0	-2,0	-1,0	1,-1

Find the unique Nash Equilibrium by iteratively eliminating the strictly dominated strategies.

3 Another Example. Finding all Equilibria

Example 5

	L	M	R
T	7,2	2,7	3,6
H	2,7	7,2	4,5
B	-4,1	1,0	5,3

No pure strategies can be eliminated by strict domination.

Best Responses

	L	M	R
T	**7**,2	2,**7**	3,6
H	2,**7**	**7**,2	4,5
B	-4,1	1,0	**5**,**3**

Pure strategies:

There exists a Nash equilibrium in pure strategies: (B, R)

Mixed strategies:

a) Of the form 3x3

1) Completely mixed. Try with T,H,B and L,M,R in the support

$u_1(T) = 7p_1 + 2p_2 + 3(1 - p_1 - p_2)$
$u_1(H) = 2p_1 + 7p_2 + 4(1 - p_1 - p_2)$
$u_1(B) = -4p_1 + 1p_2 + 5(1 - p_1 - p_2)$

$u_2(L) = 2q_1 + 7q_2 + 1(1 - q_1 - q_2)$
$u_2(M) = 7q_1 + 2q_2$
$u_2(R) = 6q_1 + 5q_2 + 3(1 - q_1 - q_2)$

$u_1(T) = u_1(H)$ implies $7p_1 + 2p_2 + 3(1 - p_1 - p_2) = 2p_1 + 7p_2 + 4(1 - p_1 - p_2)$ which implies $-5p_1 + 5p_2 + (1 - p_1 - p_2) = 0$. This results in $6p_1 - 4p_2 = 1$. Solving for p_1 we get $p_1 = \frac{1}{6} + \frac{2}{3}p_2$.

Setting $u_1(H) = u_1(B)$, $2p_1 + 7p_2 + 4(1 - p_1 - p_2) = -4p_1 + 1p_2 + 5(1 - p_1 - p_2)$ which implies $7p_1 + 7p_2 = 1$. Substituting $p_1 = \frac{1}{6} + \frac{2}{3}p_2$ we get $\frac{7}{6} + \frac{14}{3}p_2 + 7p_2 = 1$. This implies $p_2 < 0$. Hence we have no equilibrium with this support.

b) Of the form 3x2

2) Player 1 mixes completely and player 2 uses L,M.

$u_1(T) = 7p + 2\,(1 - p)$
$u_1(H) = 2p + 7\,(1 - p)$
$u_1(B) = -4p + 1\,(1 - p)$

$u_1(T) = u_1(H)$ implies $p = 1 - p$ and therefore $p = \frac{1}{2}$. This means that

$u_1(T) = 7 \times \frac{1}{2} + 2\left(1 - \frac{1}{2}\right) = \frac{9}{2} \neq -\frac{3}{2} = -4 \times \frac{1}{2} + 1\left(1 - \frac{1}{2}\right) = u_1(B)$. Therefore, player 1 cannot be mixing T,H and B.

3) Player 1 mixes completely and player 2 uses L,R.

$u_1(T) = 7p + 3(1 - p)$
$u_1(H) = 2p + 4(1 - p)$
$u_1(B) = -4p + 5(1 - p)$

$u_1(T) = u_1(H)$ then $7p + 3(1 - p) = 2p + 4(1 - p) \Leftrightarrow 5p = (1 - p) \Leftrightarrow p = \frac{1}{6}$.

Then, $u_1(T) = 7 \times \frac{1}{6} + 3(1 - \frac{1}{6}) = \frac{22}{6}$ and $u_1(B) = -4 \times \frac{1}{6} + 5(1 - \frac{1}{6}) = \frac{21}{6}$. Therefore, player 1 cannot be mixing T,H and B.

4) Player 1 mixes completely and player 2 uses M,R.

$u_1(T) = 2p + 3(1 - p)$
$u_1(H) = 7p + 4(1 - p)$
$u_1(B) = 1p + 5(1 - p)$

$u_1(T) = u_1(H)$ then $2p + 3(1 - p) = 7p + 4(1 - p) \Leftrightarrow 5p = -(1 - p) \Leftrightarrow p = -\frac{1}{4}$ (Note that T is strictly dominated by H if player 2's strategies are restricted mixing M and R). **Therefore, player 1 cannot be mixing T and H when player 2 mixes only M and R.**

c) Of the form 2x3

5) Player 1 mixes T,H and player 2 mixes completely.

$u_2(L) = 2q + 7(1-q)$
$u_2(M) = 7q + 2(1-q)$
$u_2(R) = 6q + 5(1-q)$

$u_2(L) = u_2(M)$ then $q = 1 - q \Leftrightarrow q = \frac{1}{2}$. With $q = \frac{1}{2}$, $u_2(L) = 2 \times \frac{1}{2} + 7\left(1 - \frac{1}{2}\right) = \frac{9}{2}$ and $u_2(R) = 6 \times \frac{1}{2} + 5\left(1 - \frac{1}{2}\right) = \frac{11}{2}$. Therefore, player 2 won't mix.

6) Player 1 mixes T,B and player 2 mixes completely.

$u_2(L) = 2q + (1-q)$
$u_2(M) = 7q$
$u_2(R) = 6q + 3(1-q)$

$u_2(L) = u_2(M)$ then $2q + (1-q) = 7q \Leftrightarrow 5q = 1 - q \Leftrightarrow q = \frac{1}{6}$. This implies $u_2(M) = 7 \times \frac{1}{6} = \frac{7}{6}$ and $u_2(R) = 6 \times \frac{1}{6} + 3\left(1 - \frac{1}{6}\right) = \frac{21}{6}$. Therefore, player 2 won't mix.

7) Player 1 mixes H,B and player 2 mixes completely.

$u_2(L) = 7q + (1-q)$
$u_2(M) = 2q$
$u_2(R) = 5q + 3(1-q)$

$u_2(L) = u_2(M)$ then $7q + (1-q) = 2q \Leftrightarrow 5q = -(1-q) \Leftrightarrow q = -\frac{1}{4} < 0$ (Note that M is strictly dominated by L when player 1 uses strategies that place positive probability only on H and B). **Therefore, player 2 cannot be mixing L and M when player 1 mixes only H and B**.

d) Of the form 2x2

8) Player 1 mixes T,H and player 2 mixes L,M

$u_1(T) = 7p + 2(1-p)$
$u_1(H) = 2p + 7(1-p)$
$u_2(L) = 2q + 7(1-q)$
$u_2(M) = 7q + 2(1-q)$

$u_1(T) = u_1(H)$ then $7p + 2(1-p) = 2p + 7(1-p) \Leftrightarrow p = 1 - p \Leftrightarrow p = \frac{1}{2} \Rightarrow u_1(T) = 7 \times \frac{1}{2} + 2\left(1 - \frac{1}{2}\right) = \frac{9}{2}$

$u_2(L) = u_2(M)$ then $2q + 7\left(1-q\right) = 7q + 2\left(1-q\right) \Leftrightarrow q = 1-q \Leftrightarrow q = \frac{1}{2} \Rightarrow$ $u_2(L) = 2 \times \frac{1}{2} + 7\left(1 - \frac{1}{2}\right) = \frac{9}{2}$

Now we need to check if none of the players wants to deviate from the proposed mixed strategy

$u_1(B) = -4 \times \frac{1}{2} + 1\left(1 - \frac{1}{2}\right) = -\frac{3}{2}$ Player 1 doesn't want to deviate

$u_2(R) = 6 \times \frac{1}{2} + 5\left(1 - \frac{1}{2}\right) = \frac{11}{2}$ Player 2 wants to deviate

9) Player 1 mixes T,H and player 2 mixes L,R

$u_1(T) = 7p + 3(1-p)$
$u_1(H) = 2p + 4(1-p)$

$u_2(L) = 2q + 7\left(1-q\right)$
$u_2(R) = 6q + 5\left(1-q\right)$

$u_1(T) = u_1(H)$, implies $7p + 3(1-p) = 2p + 4(1-p)$ which results in $5p = (1-p)$ and $p = \frac{1}{6}$

$u_2(L) = u_2(R)$, implies $2q + 7\left(1-q\right) = 6q + 5\left(1-q\right)$ which results in $2q = (1-q)$ and $q = \frac{1}{3}$

Payoff for 1 is $u_1(T) = 7 \times \frac{1}{6} + 3(1 - \frac{1}{6}) = \frac{22}{6}$. By using B player 1 gets $u_1(B) = -4 \times \frac{1}{6} + 5(1 - \frac{1}{6}) = \frac{21}{6}$, thus he does not wish to deviate.

Payoff for 2 is $u_2(L) = 2 \times \frac{1}{3} + 7\left(1 - \frac{1}{3}\right) = \frac{16}{3}$. By using M player 1 gets $u_1(M) = 7 \times \frac{1}{3} + 2(1 - \frac{1}{3}) = \frac{11}{3}$, thus he does not wish to deviate.

There exists a mixed strategy Nash equilibrium of the form: $\sigma_1 = \left(\frac{1}{3}, \frac{2}{3}, 0\right), \sigma_2 = \left(\frac{1}{6}, 0, \frac{5}{6}\right)$.

10) Player 1 mixes T,H and player 2 mixes M,R

No equilibrium in this case - see case (4).

11) Player 1 mixes T,B and player 2 mixes L,M

B is strictly dominated by T for player 1

12) Player 1 mixes T,B and player 2 mixes L,R

L is strictly dominated by R for player 2

13) Player 1 mixes T,B and player 2 mixes M,R

$u_1(T) = 2p + 3(1-p)$
$u_1(B) = 1p + 5(1-p)$

$u_2(M) = 7q$
$u_2(R) = 6q + 3\,(1-q)$

$u_1(T) = u_1(B)$ then $2p + 3(1-p) = p + 5(1-p) \Leftrightarrow p = 2\,(1-p) \Leftrightarrow p = \frac{2}{3}$
$u_2(M) = u_2(R)$ then $7q = 6q + 3\,(1-q) \Leftrightarrow q = 3\,(1-q) \Leftrightarrow q = \frac{3}{4}$
$u_1(T) = 2 \times \frac{2}{3} + 3(1-\frac{2}{3}) = \frac{7}{3}$
$u_2(M) = 7 \times \frac{3}{4} = \frac{21}{4}$

$u_1(H) = 7 \times \frac{2}{3} + 4(1-\frac{2}{3}) = \frac{18}{3}$ This means that player 1 prefers to deviate from the proposed mixed strategy. We could have figured this out earlier since T is strictly dominated by H for player 1 if player 2 mixes M and R.

14) Player 1 mixes H,B and player 2 mixes L,M

No equilibrium in this case - see case (7)

15) Player 1 mixes H,B and player 2 mixes L,R

$u_1(H) = 2p + 4(1-p)$
$u_1(B) = -4p + 5(1-p)$

$u_2(L) = 7q + 1\,(1-q)$
$u_2(R) = 5q + 3\,(1-q)$
$u_1(H) = u_1(B)$ then $2p+4(1-p) = -4p+5(1-p) \Leftrightarrow 6p = (1-p) \Leftrightarrow p = \frac{1}{7}$
$u_2(L) = u_2(R)$ then $7q+1\,(1-q) = 5q+3\,(1-q) \Leftrightarrow 2q = 2\,(1-q) \Leftrightarrow q = \frac{1}{2}$

$u_1(H) = 2 \times \frac{1}{7} + 4(1-\frac{1}{7}) = \frac{26}{7}$
$u_2(L) = 7 \times \frac{1}{2} + 1\left(1-\frac{1}{2}\right) = 4$

$u_1(T) = 7 \times \frac{1}{7} + 3(1-\frac{1}{7}) = \frac{25}{7}$
$u_2(M) = 2 \times \frac{1}{2} = 1$

No player wants to deviate from the proposed mixed strategy. Hence, there exists a mixed strategy Nash equilibrium of the form: $\sigma_1 = \left(0, \frac{1}{2}, \frac{1}{2}\right), \sigma_2 = \left(\frac{1}{7}, 0, \frac{6}{7}\right)$.

16) Player 1 mixes H,B and player 2 mixes M,R

M is strictly dominated by R for player 2.

e) Of the form 1xk or kx1 with $k \in \{2, 3\}$

It is easy to check that no such equilibrium exists because if we look at the payoffs of player 1 no number is repeated in a given column and if we look at player 2's payoffs no number is repeated in a given row. (Why does this imply that no such equilibrium exists?)

Summing up

The game has three Nash equilibria
$\sigma_1 = (0, 0, 1), \sigma_2 = (0, 0, 1)$
$\sigma_1 = \left(\frac{1}{3}, \frac{2}{3}, 0\right), \sigma_2 = \left(\frac{1}{6}, 0, \frac{5}{6}\right)$
$\sigma_1 = \left(0, \frac{1}{2}, \frac{1}{2}\right), \sigma_2 = \left(\frac{1}{7}, 0, \frac{6}{7}\right)$