Notes on the LRSR Folk Theorem
Course Econ 201B

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1 Notation and Definitions

Actions in the stage game for player i: $a^i \in A^i$ finite

Mixed actions for player i: $\alpha^i : A^i \to [0,1]$ and $\sum_{a^i \in A^i} \alpha^i\left(a^i\right) = 1$

Patience parameter: $\delta \in [0,1)$

Best and worst dynamic payoff for player 1: $\bar{v}^1$, $\underline{v}^1$

Equilibrium payoff for player 1 beginning next period if action a^1 was selected: $w^1\left(a^1\right)$. In general, we can think of the function $w\left(\cdot\right)$ as a function that maps actions into real numbers. $w : A^1 \to R$.

Static Nash equilibrium payoff to player 1:

$$n^1 = u^1\left(\alpha^1, \alpha^2\right) \text{ such that } \alpha^1 \in BR^1\left(\alpha^2\right) \wedge \alpha^2 \in BR^2\left(\alpha^1\right)$$

Pure Stackelberg payoff to player 1:

$$ps^1 = \max_{(a^1,\alpha^2):\alpha^2 \in BR^2(a^1)} u^1\left(a^1, \alpha^2\right)$$

Mixed Stackelberg payoff to player 1:

$$ms^1 = \max_{(\alpha^1,\alpha^2):\alpha^2 \in BR^2(\alpha^1)} \sum_{a^1} u^1\left(a^1, \alpha^2\right) \alpha^1\left(a^1\right)$$

Minmax payoff to player 1:

$$m^1 = \min_{\alpha^2} \max_{a^1} u^1\left(a^1, \alpha^2\right)$$

2 Characterization of equilibrium payoffs

For v^1 to be the equilibrium payoff associated to a fixed α^1, the following two equations have to be satisfied

$$v^1 \geq (1-\delta) u^1 \left(a^1, \alpha^2\right) + \delta w^1 \left(a^1\right), \qquad \forall a^1 \in A^1 \tag{IC}$$

and

$$v^1 = (1-\delta) u^1 \left(a^1, \alpha^2\right) + \delta w^1 \left(a^1\right), \qquad \forall a^1 : \alpha^1 \left(a^1\right) > 0 \tag{MIX}$$

Additionally, the continuation payoffs have to be sustainable

$$\underline{v}^1 \leq w^1 \left(a^1\right) \leq \bar{v}^1, \qquad \forall a^1 \in A^1 \tag{AG}$$

2.1 Characterization of the lowest and highest payoff

To find $\underline{v}^1$ and $\bar{v}^1$ the following problem has to be solved:

Choose $\alpha = \left(\alpha^1, \alpha^2\right)$, and $w^1 \left(a^1\right)$ for all $a^1 \in A^1$ such that $\underline{v}^1$ is minimized and $\bar{v}^1$ is maximized subject to the following constraints

$$\alpha^2 \in BR^2 \left(\alpha^1\right) \tag{BR}$$

$$\bar{v}^1 \geq (1-\delta) u^1 \left(a^1, \alpha^2\right) + \delta w^1 \left(a^1\right), \qquad \forall a^1 \in A^1 \tag{IC$_1$}$$

$$\bar{v}^1 = (1-\delta) u^1 \left(a^1, \alpha^2\right) + \delta w^1 \left(a^1\right), \qquad \forall a^1 : \alpha^1 \left(a^1\right) > 0 \tag{MIX$_1$}$$

$$\underline{v}^1 \geq (1-\delta) u^1 \left(a^1, \alpha^2\right) + \delta w^1 \left(a^1\right), \qquad \forall a^1 \in A^1 \tag{IC$_2$}$$

$$\underline{v}^1 = (1-\delta) u^1 \left(a^1, \alpha^2\right) + \delta w^1 \left(a^1\right), \qquad \forall a^1 : \alpha^1 \left(a^1\right) > 0 \tag{MIX$_2$}$$

$$\underline{v}^1 \leq w^1 \left(a^1\right) \leq \bar{v}^1, \qquad \forall a^1 \in A^1 \tag{AG}$$

Note that something is missing in this problem: the weights we assign to the minimization and maximization parts. In principle, there could be a trade-off between these two parts of the problem. Fortunately, this is not the case. In fact, for large enough values of δ the problem can be separated into a maximization problem and a minimization problem. To do that we need to partition constraint (AG) into two separate constraints, since this constraint involves both $\underline{v}^1$ and

$\bar{v}^1$. To do that we use the value of the (a) static Nash equilibrium n^1. The constraints we are going to use are

$$n^1 \leq w^1\left(a^1\right) \leq \bar{v}^1, \qquad \forall a^1 \in A^1 \tag{AG$_1$}$$

$$\underline{v}^1 \leq w^1\left(a^1\right) \leq n^1, \qquad \forall a^1 \in A^1 \tag{AG$_2$}$$

Note that these two constraints are tighter than (AG). For example, no single $w^1(\cdot)$ can satisfy both at the same time unless $w^1\left(a^1\right) = n^1$ for all $a^1 \in A^1$. By imposing constraints that are tighter than (AG) we are being conservative in our calculation of $\underline{v}^1$ and $\bar{v}^1$. But it can be shown that as $\delta \to 1$, $w^1\left(a^1\right)$ can be chosen as close to $\underline{v}^1$ and $\bar{v}^1$ as desired, so this being conservative is only apparent.

The two separate problems are stated below.

MAX PROBLEM

$$\max_{(\alpha^1,\alpha^2),w^1(\cdot)} \bar{v}^1 \tag{1}$$

subject to

$$\alpha^2 \in BR^2\left(\alpha^1\right) \tag{BR}$$

$$\bar{v}^1 \geq (1-\delta)\, u^1\left(a^1, \alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall a^1 \in A^1 \tag{IC$_1$}$$

$$\bar{v}^1 = (1-\delta)\, u^1\left(a^1, \alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall a^1 : \alpha^1\left(a^1\right) > 0 \tag{MIX$_1$}$$

$$n^1 \leq w^1\left(a^1\right) \leq \bar{v}^1, \qquad \forall a^1 \in A^1 \tag{AG$_1$}$$

MIN PROBLEM

$$\min_{(\alpha^1,\alpha^2),w^1(\cdot)} \underline{v}^1 \tag{2}$$

$$\alpha^2 \in BR^2\left(\alpha^1\right) \tag{BR}$$

$$\underline{v}^1 \geq (1-\delta)\, u^1\left(a^1, \alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall a^1 \in A^1 \tag{IC$_2$}$$

$$\underline{v}^1 = (1-\delta)\, u^1\left(a^1, \alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall \alpha^1\left(a^1\right) > 0 \tag{MIX$_2$}$$

$$\underline{v}^1 \leq w^1\left(a^1\right) \leq n^1, \qquad \forall a^1 \in A^1 \tag{AG$_2$}$$

2.2 Solving for $\bar{v}^1$

As is usual when maximizing stuff with lots of constraints, we will ignore some constraints and show later that the solution we found satisfies them anyway. The constraints we will momentarily forget about are conditions (IC_1) and a part of (AG_1). We will check later that they are satisfied. (IC_1) will disappear completely and we will only consider the following partial version of (AG_1)

$$w^1\left(a^1\right) \leq \bar{v}^1, \qquad \forall a^1 \in A^1 \tag{AG'$_1$}$$

Note that only (BR) involves the actual mixed strategies by player 1. All other constraints involve pure strategies. It is useful to rewrite the maximization problem as a two step maximization

$$\max_{(\alpha^1,\alpha^2):\alpha^2\in BR^2(\alpha^1)} \max_{w^1(\cdot)} \bar{v}^1 \tag{3}$$

subject to

$$\bar{v}^1 = (1-\delta)\, u^1\left(a^1, \alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall a^1 : \alpha^1\left(a^1\right) > 0 \tag{MIX$_1$}$$

$$w^1\left(a^1\right) \leq \bar{v}^1, \qquad \forall a^1 \in A^1 \tag{AG'$_1$}$$

The inner maximization therefore takes $\left(\alpha^1, \alpha^2\right)$ as given and chooses the function $w^1(\cdot)$ so as to maximize $\bar{v}^1$. In a second step $\left(\alpha^1, \alpha^2\right)$ are chosen.

2.2.1 Inner maximization for a fixed $\left(\alpha^1, \alpha^2\right)$. Choosing $w(\cdot)$

The inner maximization is

$$\max_{w^1(\cdot)} \bar{v}^1 \tag{4}$$

subject to

$$\bar{v}^1 = (1-\delta)\, u^1\left(a^1, \alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall a^1 : \alpha^1\left(a^1\right) > 0 \tag{MIX$_1$}$$

$$w^1\left(a^1\right) \leq \bar{v}^1, \qquad \forall a^1 \in A^1 \tag{AG'$_1$}$$

$$(\alpha_1, \alpha_2) \text{ given}$$

Now consider constraint (MIX_1). If α_1 and α_2 are fixed then the only thing that helps in making the payoff $\bar{v}^1$ as high as possible is choosing $w^1\left(a^1\right)$ as large as possible. However, if the fixed α_1 is a mixed strategy (i.e. it puts positive probability on more than one pure action), then this constraint requires that the highest $w^1\left(a^1\right)$ goes together with the smallest $u^1\left(a^1,\alpha^2\right)$. From (AG_1'), the highest value we are allowed to choose for $w^1\left(a^1\right)$ is $\bar{v}^1$. Calculate $\tilde{a}^1$ as

$$\tilde{a}^1 = \arg \min_{a^1:\alpha^1(a^1)>0} u^1\left(a^1,\alpha^2\right). \tag{5}$$

For that value of $\tilde{a}^1$ we are allowed to choose $w^1\left(a^1\right) = \bar{v}^1$ and condition (MIX_1) becomes

$$\bar{v}^1 = (1-\delta) \min_{a^1:\alpha^1(a^1)>0} u^1\left(a^1,\alpha^2\right) + \delta\bar{v}^1 \tag{6}$$

Therefore, the result for the inner maximization is

$$\bar{v}^1_{(\alpha_1,\alpha_2)} = \min_{a^1:\alpha^1(a^1)>0} u^1\left(a^1,\alpha^2\right) \tag{7}$$

This equation makes it explicit that the value we found is for a fixed (α_1,α_2) and, therefore depends on (α_1,α_2).

2.2.2 Outer maximization: choosing $\left(\alpha^1,\alpha^2\right)$

The outer maximization requires that we find $\left(\alpha^1,\alpha^2\right)$ so as to maximize the value we got for $\bar{v}^1_{(\alpha_1,\alpha_2)}$

$$\max_{(\alpha^1,\alpha^2):\alpha^2\in BR^2(\alpha^1)} \max_{w^1(\cdot)} \bar{v}^1_{(\alpha_1,\alpha_2)} = \max_{(\alpha^1,\alpha^2):\alpha^2\in BR^2(\alpha^1)} \min_{a^1:\alpha^1(a^1)>0} u^1\left(a^1,\alpha^2\right) \tag{8}$$

Therefore the solution for the largest $\bar{v}_1$ is

$$\bar{v}_1 = \max_{(\alpha^1,\alpha^2):\alpha^2\in BR^2(\alpha^1)} \min_{a^1:\alpha^1(a^1)>0} u^1\left(a^1,\alpha^2\right) \tag{9}$$

2.2.3 The ignored conditions AG$_1$ and IC$_1$

Now it is time to verify that our solution satisfies conditions AG$_1$ and IC$_1$. We already know that AG'$_1$ is satisfied, so the only thing we need for AG$_1$ is that

$$n^1 \le w^1\left(a^1\right), \qquad \forall a^1 \in A^1 \tag{10}$$

We know that $\bar{v}_1 \ge n^1$. There are two cases:
1) $\bar{v}^1 = n^1$
2) $\bar{v}^1 > n^1$

In the first case we don't have to prove anything since AG_1 holds trivially and IC_1 holds because $\bar{v}^1$ can be derived from a static Nash Equilibrium. Thus, in the remainder we focus on case 2: $\bar{v}^1 > n^1$

AG_1 is satisfied

Use the IC_1 to solve for $w^1\left(a^1\right)$

$$w^1\left(a^1\right) \leq \frac{\bar{v}^1 - (1-\delta)\, u^1\left(a^1, \alpha^2\right)}{\delta} \tag{11}$$

Now consider what happens to the RHS as $\delta \to 1$

$$\lim_{\delta \to 1} \frac{\bar{v}^1 - (1-\delta)\, u^1\left(a^1, \alpha^2\right)}{\delta} = \bar{v}^1 \tag{12}$$

Since we are considering case 2 we have that

$$\lim_{\delta \to 1} \frac{\bar{v}^1 - (1-\delta)\, u^1\left(a^1, \alpha^2\right)}{\delta} = \bar{v}^1 > n^1 \tag{13}$$

Then, for large enough δ it is always possible to choose $w^1\left(a^1\right)$ such that $w^1\left(a^1\right) \geq n^1$.

IC_1 is satisfied

What about IC_1? We are going to look for the value of δ so that the IC is satisfied no matter what the value of $w^1\left(a^1\right)$ is

$$\bar{v}^1 \geq (1-\delta)\, u^1\left(a^1, \alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall a^1 \in A^1 \tag{IC_1}$$

Since $w^1\left(a^1\right) \leq \bar{v}^1$, by AG'$_1$, this equation will automatically hold for any δ when $\bar{v}^1 \geq u^1\left(a^1, \alpha^2\right)$. Therefore, consider those a^1 for which $u^1\left(a^1, \alpha^2\right) > \bar{v}^1$. Rewrite IC_1 to get

$$\bar{v}^1 - u^1\left(a^1, \alpha^2\right) \geq \delta\left[w^1\left(a^1\right) - u^1\left(a^1, \alpha^2\right)\right], \qquad \forall a^1 \in A^1 \tag{14}$$

Since we have negative numbers on both sides it is convenient to write the inequality as

$$\delta\left[u^1\left(a^1, \alpha^2\right) - w^1\left(a^1\right)\right] \geq u^1\left(a^1, \alpha^2\right) - \bar{v}^1 \tag{15}$$

Divide by $u^1\left(a^1, \alpha^2\right) - w^1\left(a^1\right)$

$$\delta \geq \frac{u^1\left(a^1, \alpha^2\right) - \bar{v}^1}{u^1\left(a^1, \alpha^2\right) - w^1\left(a^1\right)} \tag{16}$$

Define $\bar{\delta}_{a^1}$ as the value for δ such that this equation is satisfied with equality for a specific a^1.

Then take

$$\bar{\delta} = \max_{a^1 \in A^1} \bar{\delta}_{a^1} \tag{17}$$

Thus, for $\delta \geq \bar{\delta}$ the IC_1 is satisfied.

2.3 Solving for $\underline{v}^1$

We have to solve the minimization problem. The strategy will be the same as for the maximization problem above, proceeding in two steps. The only difference is that we also include IC_2 into the inner minimization.

$$\min_{(\alpha^1,\alpha^2):\alpha^2\in BR^2(\alpha^1)} \min_{w^1(\cdot)} \underline{v}^1$$

subject to

$$\underline{v}^1 \geq (1-\delta)\, u^1\left(a^1,\alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall a^1 \in A^1 \tag{IC$_2$}$$

$$\underline{v}^1 = (1-\delta)\, u^1\left(a^1,\alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall \alpha^1\left(a^1\right) > 0 \tag{MIX$_2$}$$

$$\underline{v}^1 \leq w^1\left(a^1\right), \qquad \forall a^1 \in A^1 \tag{AG'$_2$}$$

2.3.1 Inner minimization

$$\min_{w^1(\cdot)} \underline{v}^1$$

subject to

$$\underline{v}^1 \geq (1-\delta)\, u^1\left(a^1,\alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall a^1 \in A^1 \tag{IC$_2$}$$

$$\underline{v}^1 = (1-\delta)\, u^1\left(a^1,\alpha^2\right) + \delta w^1\left(a^1\right), \qquad \forall \alpha^1\left(a^1\right) > 0 \tag{MIX$_2$}$$

$$\underline{v}^1 \leq w^1\left(a^1\right), \qquad \forall a^1 \in A^1 \tag{AG'$_2$}$$

$$(\alpha_1, \alpha_2) \text{ given}$$

We want $w^1\left(a^1\right)$ to be as small as possible. By AG'$_2$ the smallest choice for $w^1\left(a^1\right)$ is $\underline{v}^1$. We want the RHS of IC_2 to be as small as possible. This equation will bind for the first time when $u^1\left(a^1,\alpha^2\right)$ is biggest.

$$\underline{v}^1 = (1-\delta) \max_{a^1} u^1\left(a^1,\alpha^2\right) + \delta \underline{v}^1 \tag{18}$$

Therefore,

$$\underline{v}^1_{(\alpha^1,\alpha^2)} = \max_{a^1} u^1\left(a^1,\alpha^2\right) \tag{19}$$

2.3.2 Outer minimization

By the outer minimization

$$\underline{v}^1 = \min_{(\alpha^1,\alpha^2):\alpha^2 \in BR^2(\alpha^1)} \underline{v}^1_{(\alpha^1,\alpha^2)} \tag{20}$$

Using the result of the inner minimization

$$\underline{v}^1 = \min_{(\alpha^1,\alpha^2):\alpha^2 \in BR^2(\alpha^1)} \max_{a^1} u^1\left(a^1,\alpha^2\right) \tag{21}$$

AG_2 is satisfied by a similar argument as the one we had before.

2.4 Set of equilibrium payoffs

By virtue of public randomization, the set of equilibrium payoffs is convex. Therefore, by finding $\bar{v}^1$ and $\underline{v}^1$ we already know what the entire set looks like. Let's now think about how $\bar{v}^1$ and $\underline{v}^1$ compare to other values we can calculate from the static game.

2.4.1 $\bar{v}^1$ vs Pure and Mixed Stackelberg

The relationship is

$$ps^1 \leq \bar{v}^1 \leq ms^1 \tag{22}$$

Start with the definition of $\bar{v}_1$

$$\bar{v}_1 = \max_{(\alpha^1,\alpha^2):\alpha^2 \in BR^2(\alpha^1)} \min_{a^1:\alpha^1(a^1)>0} u^1\left(a^1,\alpha^2\right) \tag{23}$$

Now impose the additional constraint that α^1 is a pure strategy, i.e. it has to put positive probability on only one element of A^1. By adding an additional constraint this calculation can only go down. Therefore,

$$\bar{v}_1 \geq \max_{(a^1,\alpha^2):\alpha^2 \in BR^2(a^1)} \min_{a^1} u^1\left(a^1,\alpha^2\right) \tag{24}$$

The inner minimization is over a single point. Thus,

$$\bar{v}_1 \geq \max_{(a^1,\alpha^2):\alpha^2 \in BR^2(a^1)} u^1\left(a^1,\alpha^2\right) = ps^1 \tag{25}$$

In consequence,

$$ps^1 \leq \bar{v}_1 \tag{26}$$

Now consider the definition of the mixed Stackelberg payoff

$$ms^1 = \max_{(\alpha^1,\alpha^2):\alpha^2 \in BR^2(\alpha^1)} \sum_{a^1} u^1\left(a^1,\alpha^2\right)\alpha^1\left(a^1\right) \tag{27}$$

When compared to $\bar{v}_1$, it is evident that the outer maximization is the same. Therefore, $ms^1 \geq \bar{v}_1$ iff

$$\sum_{a^1} u^1\left(a^1, \alpha^2\right) \alpha^1\left(a^1\right) \geq \min_{a^1 : \alpha^1(a^1) > 0} u^1\left(a^1, \alpha^2\right) \tag{28}$$

Note that terms for which $\alpha^1\left(a^1\right) = 0$ drop out of the RHS because they are multiplied by this number.

$$\sum_{a^1} u^1\left(a^1, \alpha^2\right) \alpha^1\left(a^1\right) = \sum_{a^1 : \alpha^1(a^1) > 0} u^1\left(a^1, \alpha^2\right) \alpha^1\left(a^1\right) \geq \min_{a^1 : \alpha^1(a^1) > 0} u^1\left(a^1, \alpha^2\right) \tag{29}$$

This last inequality is true because an average of a group of numbers is always at least as great as the minimum of that group of numbers. Hence,

$$\bar{v}_1 \leq ms^1 \tag{30}$$

2.4.2 $\underline{\mathrm{v}}^1$ vs Minmax and Static Nash

The relationship is

$$m^1 \leq \underline{\mathrm{v}}^1 \leq n^1 \tag{31}$$

Consider the definition of the minmax

$$m^1 = \min_{\alpha^2} \max_{a^1} u^1\left(a^1, \alpha^2\right)$$

Now additionally restrict α^2 to be a best response to α^1 in the outer minimization. By doing this the value cannot get smaller. Thus,

$$m^1 \leq \min_{(\alpha^1, \alpha^2) : \alpha^2 \in BR^2(\alpha^1)} \max_{a^1} u^1\left(a^1, \alpha^2\right) = \underline{\mathrm{v}}^1 \tag{32}$$

The comparison with the static Nash was done before. Therefore,

$$m^1 \leq \underline{\mathrm{v}}^1 \leq n^1 \tag{33}$$

3 Examples

This section calculates all the above values in examples. For convenience, all the formulas are grouped together in the following table

Pure Stackelberg	$ps^1 = \max_{(a^1,\alpha^2):\alpha^2\in BR^2(a^1)} u^1\left(a^1,\alpha^2\right)$
Mixed Stackelberg	$ms^1 = \max_{(\alpha^1,\alpha^2):\alpha^2\in BR^2(\alpha^1)} \sum_{a^1} u^1\left(a^1,\alpha^2\right)\alpha^1\left(a^1\right)$
Minmax	$m^1 = \min_{\alpha^2}\max_{a^1} u^1\left(a^1,\alpha^2\right)$
Best dynamic payoff	$\bar{v}_1 = \max_{(\alpha^1,\alpha^2):\alpha^2\in BR^2(\alpha^1)} \min_{a^1:\alpha^1(a^1)>0} u^1\left(a^1,\alpha^2\right)$
Worst dynamic payoff	$\underline{v}^1 = \min_{(\alpha^1,\alpha^2):\alpha^2\in BR^2(\alpha^1)} \max_{a^1} u^1\left(a^1,\alpha^2\right)$

In all examples the row player is the long run player and the column player is a short run player.

3.1 First Example

Consider the following stage game.

	L	R
U	2,0	-1,-1
D	3,0	1,1

Static Nash

The static Nash Equilibrium is (D,R) and yields a payoff $n^1 = 1$
Pure Stackelberg

Action	BR^2	payoff
U	L	2
D	R	1

$$ps^1 = \max\{1,2\} = 2$$

Mixed Stackelberg

Action	BR^2	payoff
U or mix $p > \frac{1}{2}$	L	2
mix $p = \frac{1}{2}$	L, R, mix	2, 1, $[0, 2.5]$
D or mix $p < \frac{1}{2}$	R	1

$$ms^1 = \max\{1, 2, [0, 2.5]\} = 2.5$$

Minmax

Action	BR^2	payoff
L	D	3
mix	D	$[1, 3]$
R	D	1

$$m^1 = \min\{3, [1,3], 1\} = 1$$

Best dynamic payoff

Action	BR^2	worst in support
U or mix $p > \frac{1}{2}$	L	2
mix $p = \frac{1}{2}$	L, R, mix	2, -1, $[-1, 2]$
D or mix $p < \frac{1}{2}$	R	1

$$\bar{v}_1 = \max\{1, 2, [-1, 2]\} = 2$$

Worst dynamic payoff

Since $n^1 = m^1 = 1$ and $m^1 \leq \underline{v}^1 \leq n^1$ it has to be that $\underline{v}^1 = 1$. It is equal to the minmax because there are no dominated actions for player 2.

Summing up,

Mixed Stackelberg	$ms^1 = 2.5$
Best dynamic payoff	$\bar{v}_1 = 2$
Pure Stackelberg	$ps^1 = 2$
Static Nash	$n^1 = 1$
Worst dynamic payoff	$\underline{v}^1 = 1$
Minmax	$m^1 = 1$

Find δ

To support $\bar{v}_1 = 2$ the following IC_1 has to hold

$$2 \geq (1 - \delta) 3 + \delta 1$$

which implies

$$\delta \geq \bar{\delta} = \frac{1}{2}$$

Which strategy profile supports $\bar{v}_1 = 2$ for $\delta \geq \bar{\delta}$?

Play (U, L). If anyone ever deviates play (D, R) forever.

Which strategy profile supports $\underline{v}^1 = 1$ for $\delta \geq \bar{\delta}$?

Play (D, R) always. This is an equilibrium for all discount factors.

3.2 Second Example (Final 2001)

Consider the following stage game.

	L	R
U	0,0	3,2
D	2,3	1,1

Static Nash

There are three static Nash Equilibria: $(U, R), (D, L)$ and a mixed equilibrium $\left(\frac{1}{2}, \frac{1}{2}\right)$. This last equilibrium yields the lowest payoff $n^1 = 1.5$

Pure Stackelberg

Action	BR^2	payoff
U	R	3
D	L	2

$$ps^1 = \max\{2, 3\} = 3$$

Mixed Stackelberg

Action	BR^2	payoff
U or mix $p > \frac{1}{2}$	R	3
mix $p = \frac{1}{2}$	L, R, mix	1, 2, $[1, 2]$
D or mix $p < \frac{1}{2}$	L	2

$$ms^1 = \max\{3, 2, [1, 2]\} = 3$$

Minmax

Action	BR^2	payoff
L or mix $q > \frac{1}{2}$	D	$(1.5, 2]$
mix $q = \frac{1}{2}$	U,D,mix	1.5
R or mix $q < \frac{1}{2}$	U	$(1.5, 3]$

$$m^1 = \min\{(1.5, 2], (1.5, 3], 1.5\} = 1.5$$

Best dynamic payoff

Since

$$3 = ps^1 \leq \bar{v}^1 \leq ms^1 = 3$$

$$\bar{v}_1 = 3$$

Worst dynamic payoff
Since

$$1.5 = m^1 \leq \underline{v}^1 \leq n^1 = 1.5$$

$$\underline{v}^1 = 1.5$$

Summing up,

Mixed Stackelberg	$ms^1 = 3$
Best dynamic payoff	$\bar{v}_1 = 3$
Pure Stackelberg	$ps^1 = 3$
Static Nash	$n^1 = 1.5$
Worst dynamic payoff	$\underline{v}^1 = 1.5$
Minmax	$m^1 = 1.5$

Find δ

To support $\bar{v}_1 = 3$, δ can be anything since it can be obtained by a Nash Equilibrium of the static game. Therefore, $\bar{\delta} = 0$.

Which strategy profile supports $\bar{v}_1 = 3$ for $\delta \geq \bar{\delta}$?

Play (U, R) always.

Which strategy profile supports $\underline{v}^1 = 1.5$ for $\delta \geq \bar{\delta}$?

Play $\left(\frac{1}{2}, \frac{1}{2}\right)$ always.

3.3 Third Example (Final 2003)

Consider the following stage game.

	L	M	R	S
U	1,1	5,4	1,**5**	**0**,0
D	**3**,**5**	**6**,4	**2**,1	**0**,0

Static Nash

(D, L) is the unique static Nash equilibrium (the only action profile that survives iterated elimination of strictly dominated actions), $n^1 = 3$

Pure Stackelberg

Action	BR^2	payoff
U	R	1
D	L	3

$$ps^1 = \max\{1, 3\} = 3$$

Mixed Stackelberg

Action	BR^2	payoff
U or mix $p > \frac{3}{4}$	R	$[1, 1.25)$
$p = \frac{3}{4}$	any mix of R and M (including pure)	$[1.25, 5.25]$
mix $p \in \left(\frac{1}{4}, \frac{3}{4}\right)$	M	$(5.25, 5.75)$
$p = \frac{1}{4}$	any mix of L and M (including pure)	$[2.5, 5.75]$
D or mix $p \leq \frac{1}{4}$	L	$(2.5, 3]$

$$ms^1 = \max\{[1, 1.25), [1.25, 5.25], (5.25, 5.75), [2.5, 5.75], (2.5, 3]\} = 5.75$$

Minmax

The minmax is clearly when player 2 plays S and forces player 1 to accept a payoff of 0. The normal form of the game has the best responses on bold. Using these values to calculate m^1

$$m^1 = \min\{3, 6, 2, 0\} = 0$$

Since 0 is the lowest payoff in the matrix we do not have to worry about any mixed strategy giving player 1 a lower payoff.

Best dynamic payoff

Action	BR^2	worst in support
U or mix $p > \frac{3}{4}$	R	1
$p = \frac{3}{4}$	R, M, mix	$1, 5, (1, 5)$
mix $p \in \left(\frac{1}{4}, \frac{3}{4}\right)$	M	5
$p = \frac{1}{4}$	L, M, mix	$1, 5, (1, 5)$
D or mix $p \leq \frac{1}{4}$	L	1

$$\bar{v}^1 = \max\{1, 5, (1, 5)\} = 5$$

Worst dynamic payoff
Since

$$0 = m^1 < \underline{v}^1 < n^1 = 3$$

we actually need to do a calculation here. $\underline{v}^1$ is the "constrained" minmax, where "constrained" means that we don't allow the column player to use dominated actions when minmaxing player 1. Thus, we get rid of S because it is strictly dominated by everything else. We can't get rid of any other pure strategies because L, M, R are best responses to something. If we consider only pure strategies for player 2 then

$$\underline{v}^1 = \min\{3, 6, 2\} = 2$$

Why don't mixed strategies help to get a lower payoff in this case? It is because player 1 has a strictly dominating strategy in D once S is removed. Thus, he is assured to get at least 2 no matter what player 2 does.

Summing up,

Mixed Stackelberg	$ms^1 = 5.75$
Best dynamic payoff	$\bar{v}_1 = 5$
Pure Stackelberg	$ps^1 = 3$
Static Nash	$n^1 = 3$
Worst dynamic payoff	$\underline{v}^1 = 2$
Minmax	$m^1 = 0$

Find δ

To support $\bar{v}_1 = 5$, with a $\underline{v}^1$ threat, the incentive compatibility constraint for player 1 requires

$$5 \geq (1-\delta)\,6 + \delta 2$$

Hence,

$$\delta \geq \bar{\delta} = \frac{1}{4}$$

With Nash threats we get $\bar{\delta} = \frac{1}{3}$.

Which strategy profile supports $\bar{v}_1 = 5$ for $\delta \geq \bar{\delta} = \frac{1}{3}$?

Play $\left(\frac{1}{2}, \frac{1}{2}\right)$ and M. If the resulting play is (U, M) start over. If the result is (D, M) start over with probability π and play static Nash with probability $1-\pi$, where π is such that player 1 gets a payoff of 5 even though he temporarily gets 6, i.e.

$$5 = (1-\delta)\,6 + \delta\,[\pi 5 + (1-\pi)\,3]$$

$$\pi = \frac{3\delta - 1}{2\delta}$$

If anybody deviates play static Nash forever. SR does not deviate because he is playing a best response. LR does not deviate because IC_1 is satisfied.