Notes on Sequential Equilibrium
Course Econ 201B

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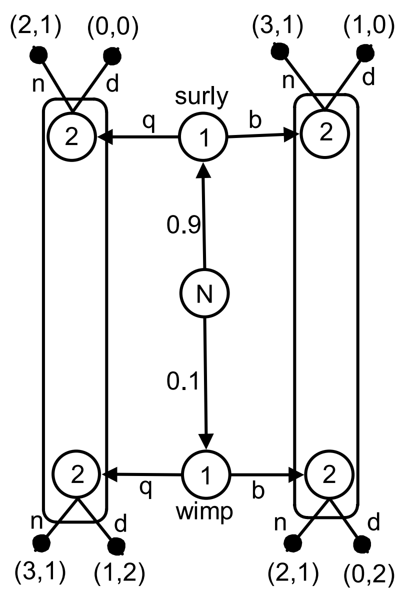
1 Notation

It is useful to consider an example. I will use the Beer-Quiche game from lecture.

1.1 Beer-Quiche example



Beer-Quiche game

1.2 Assessments

An assessment α_i is a probability distribution over nodes within an information set.[1] If we want to make specific reference to an information set h_i we can denote it $\alpha_i(h_i)$. Intuitively, an assessment is the probability distribution a player uses to calculate expected utility, and decide which is the optimal action at each information set. In the example, an assessment for player 2 is a vector of 4 elements, $\alpha_2 = (\alpha_2(s|q), \alpha_2(w|q), \alpha_2(s|b), \alpha_2(w|b))$ where $\alpha_2(s|q)+\alpha_2(w|q) = 1$ and $\alpha_2(s|b) + \alpha_2(w|b) = 1$.

1.3 Expectations of opponents strategies

Expectations of opponents strategies are what we used to call beliefs when we defined the notation for self-confirming equilibrium.[2] It is the expected behavior strategy of the opponents. The notation $\mu_i(h_{-i})$, $h_{-i} \in H_{-i}$ is the expected behavior strategy for the opponent who plays at information set h_{-i}. We can collect the expectations for all information sets belonging to the opponents in a single vector$(\mu_i(h_{-i}))_{h_{-i}\in H_{-i}}$. In the example, expectations are as follows. Player 1 has expectations $\mu_1(q) = (\mu_1(n|q), \mu_1(d|q))$ with $\mu_1(n|q) + \mu_1(d|q) = 1$ at the information set where player 2 has seen quiche, and $\mu_1(b) = (\mu_1(n|b), \mu_1(d|b))$ with $\mu_1(n|b) + \mu_1(d|b) = 1$ at the information set where player 2 has seen beer. We can collect all this information in a single vector $\mu_1 = (\mu_1(q), \mu_1(b)) = (\mu_1(n|q), \mu_1(d|q), \mu_1(n|b), \mu_1(d|b))$. Player 2 has expectations $\mu_2(s) = (\mu_2(q|s), \mu_2(b|s))$ with $\mu_2(q|s) + \mu_2(b|s) = 1$ at the information set (node) where player 1 is surly, and $\mu_2(w) = (\mu_2(q|w), \mu_2(b|w))$ with $\mu_2(q|w) + \mu_2(b|w) = 1$ at the information set (node) where player 1 is a wimp.

1.4 Beliefs

If we bundle together assessments and expectations of opponents strategies, then we have beliefs. A belief for player i is defined as $b_i = (\alpha_i, \mu_i)$. Therefore, beliefs consist of everything of which a player is not sure of: the probabilities of being at a certain node in an information set, and the play of the other players at their information sets.

[1] I use the notation α_i instead of a_i because the latter was already used as notation for an action by player i.

[2] For this reason, I have decided to use the same notation as in the Notes on SCE. The class slides use the notation π, which I will keep on using for behavior strategies.

1.5 Consistent beliefs

Beliefs cannot be just anything. We will require that they make sense. The formal requirement is that they are consistent, which means that assessments can be constructed from the expected play of opponents in a reasonable way. The requirement is that beliefs are constructed by using Bayes Rule. Let x be any node in an information set in H_i and denote as $h_i(x)$ the information set in which x is contained. Then assessments have to be constructed from μ_i as

$$\alpha_i(x|h_i(x)) = \frac{\Pr(x)}{\sum_{y\in h_i(x)} \Pr(y)} \tag{1}$$

where $\Pr(y)$ is the probability of reaching node y under expected play μ_i (and also taking into account actual play by player i if he plays in a previous stage of the game, and the probabilities of Nature).

The problem that still remains is that Bayes Rule does not work at information sets that are reached with zero probability (because the denominator of (1) is zero). To solve this problem we use the notion of consistency.[3]

Definition 1 *A belief $b_i = (\alpha_i, \mu_i)$ is consistent if $\alpha_i = \lim \alpha_i^n$, where α_i^n is the assessment that was constructed by using Bayes Rule on a strictly positive sequence $\mu_i^n \to \mu_i$.*

1.6 Utilities

Once we have beliefs we can talk about expected utilities, calculated according to the distribution over terminal nodes induced by a behavior strategy π_i and beliefs $b_i = (\alpha_i, \mu_i)$.

$$u_i(\pi_i|b_i) = \sum_z u_i(z|\pi_i, \mu_i) \Pr(z|\alpha_i, \pi_i, \mu_i) \tag{2}$$

We can also define expected utilities for player i conditional on a given information set h_i being reached. Define $\pi_i^{h_i}$ and $b_i^{h_i} = \left(\alpha_i^{h_i}, \mu_i^{h_i}\right)$ as the restrictions of π_i, $b_i = (\alpha_i, \mu_i)$ to all information sets that follow h_i (and including h_i). We start the calculation at information set h_i and consider only those elements of the behavior strategies and beliefs that are relevant for information sets that might follow from h_i. Utility conditional on reaching h_i is then defined as

$$u_i^{h_i}(\pi_i|b_i) \equiv \sum_z u_i\left(z|\pi_i^{h_i}, \mu_i^{h_i}\right) \Pr\left(z|\alpha_i^{h_i}, \pi_i^{h_i}, \mu_i^{h_i}\right) \tag{3}$$

[3] Consistency already implies the use of Bayes Rule at nodes that are reached with positive probability.

2 Sequential Equilibrium

2.1 Sequential rationality

Sequential rationality means that every player is playing optimally (i.e. does not wish to deviate) at all his information sets given his beliefs (i.e. assessments and expectations of opponents' play).

Definition 2 *Behavior strategy π_i is sequentially rational for player i if there does not exist a profitable deviation $\pi_i' \in \Pi_i$, given beliefs b_i for player i at any of i's information sets, i.e.*

$$u_i^{h_i}(\pi_i|b_i) \geq u_i^{h_i}(\pi_i'|b_i), \forall \pi_i' \in \Pi_i, \forall h_i \in H_i \tag{4}$$

2.2 Definition of Sequential Equilibrium

Definition 3 *A sequential equilibrium is a behavior strategy profile π and an assessment α_i for every player i such that (α_i, π_{-i}) is consistent and π_i is sequentially rational for every player i.*

Notice that in this definition the object of an equilibrium does not only include a (behavior) strategy profile, but an assessment for every player, as well. The definition implies that the expectation about opponents' play has to be correct, since μ_i is substituted by π_{-i}, the actual strategy profile, in player i's beliefs.

3 Example

Let's find sequential equilibria for the example. The way to find them all is conjecturing that some behavior strategy profile is an equilibrium and then checking if we can find beliefs that make it an equilibrium. I'll first consider separating equilibria, then pooling equilibria.

3.1 Are there Separating Equilibria?

3.1.1 Wimp eats quiche, beer for the surly type

Check the following profile: $\pi_1 = (\pi_1(q|s), \pi_1(b|s), \pi_1(q|w), \pi_1(b|w)) = (0,1,1,0)$.
By Bayes Rule the assessment for player 2 has to be $\alpha_2 = (\alpha_2(s|q), \alpha_2(w|q), \alpha_2(s|b), \alpha_2(w|b)) = (0,1,1,0)$. Then, player 2 has to play d at the first information set and n at the second information set, i.e. $\pi_2 = (\pi_2(n|q), \pi_2(d|q), \pi_2(n|b), \pi_2(d|b)) = (0,1,1,0)$. Now we have to check if player 1 wants to play π_1 when his expectations are that $\mu_1 = \pi_2 = (0,1,1,0)$. The answer is no. He can gain by deviating to $\pi_1' = (0,1,0,1)$ because the wimp does not want to get beaten up.

3.1.2 Wimp drinks beer, surly type eats quiche

Check the following profile: $\pi_1 = (\pi_1(q|s), \pi_1(b|s), \pi_1(q|w), \pi_1(b|w)) = (1,0,0,1)$. By Bayes Rule the assessment for player 2 has to be $\alpha_2 = (\alpha_2(s|q), \alpha_2(w|q), \alpha_2(s|b), \alpha_2(w|b)) = (1,0,0,1)$. Then, player 2 has to play n at the first information set and d at the second information set, i.e. $\pi_2 = (\pi_2(n|q), \pi_2(d|q), \pi_2(n|b), \pi_2(d|b)) = (1,0,0,1)$. Now we have to check if player 1 wants to play π_1 when his expectations are that $\mu_1 = \pi_2 = (1,0,0,1)$. Again, the answer is no. He can gain by deviating to $\pi_1' = (1,0,1,0)$ because the wimp does not want to get beaten up.

3.2 Are there Pooling Equilibria?

3.2.1 Both eat quiche

Check the following profile: $\pi_1 = (\pi_1(q|s), \pi_1(b|s), \pi_1(q|w), \pi_1(b|w)) = (1,0,1,0)$. By Bayes Rule the assessment for player 2 at the first information set has to be

$$\alpha_2(s|q) = \frac{\pi_1(q|s)\Pr(s)}{\pi_1(q|s)\Pr(s) + \pi_1(q|w)\Pr(w)} = 1 - \alpha_2(w|q) \tag{5}$$

$$\alpha_2(s|q) = \frac{1 \times 0.9}{1 \times 0.9 + 1 \times 0.1} = 0.9 = 1 - \alpha_2(w|q) \tag{6}$$

$\alpha_2 = (\alpha_2(s|q), \alpha_2(w|q), \alpha_2(s|b), \alpha_2(w|b)) = (0.9, 0.1, x, 1-x)$ with x still undetermined. Then, player 2 has to play n at the first information set. At the second information set player 2 will play n or d depending on how likely he thinks that he is facing the wimp. In fact, if the assessment is such that $\alpha_2(w|b) \geq \frac{1}{2}$ he will be willing to duel. Can such an assessment be constructed? The question is if we can construct $\alpha_2^n(w|b)$ from $\mu_2^n >> 0$ with $\mu_2^n \to (1,0,1,0)$ such that $\lim \alpha_2^n(w|b) \geq \frac{1}{2}$. Try to get

$$\alpha_2^n(w|b) = \frac{\mu_2^n(b|w)\Pr(w)}{\mu_2^n(b|s)\Pr(s) + \mu_2^n(b|w)\Pr(w)} > \frac{1}{2} \tag{7}$$

$$\alpha_2^n(w|b) = \frac{\mu_2^n(b|w) \times 0.1}{\mu_2^n(b|s) \times 0.9 + \mu_2^n(b|w) \times 0.1} > \frac{1}{2} \tag{8}$$

This requires

$$\frac{0.1}{\frac{\mu_2^n(b|s)}{\mu_2^n(b|w)} \times 0.9 + 0.1} > \frac{1}{2} \tag{9}$$

$$\frac{\mu_2^n(b|s)}{\mu_2^n(b|w)} < \frac{1}{9} \tag{10}$$

Therefore, what we need to construct such an assessment is that $\mu_2^n(b|w) > 9\mu_2^n(b|s)$, which we can always do since we allowed to construct any sequence $\mu_2^n >> 0$.

Consider then, $\alpha_2 = (0.9, 0.1, x, 1-x)$ with $x \leq \frac{1}{2}$. Then, $\pi_2 = (\pi_2(n|q), \pi_2(d|q), \pi_2(n|b), \pi_2(d|b)) = (1, 0, 0, 1)$ is sequentially rational.

Finally, we have to check if player 1 wants to play π_1 when his expectations are that $\mu_1 = \pi_2 = (1, 0, 0, 1)$. The answer is yes. By deviating he will get beaten up.

3.2.2 Both drink beer

Check the following profile: $\pi_1 = (\pi_1(q|s), \pi_1(b|s), \pi_1(q|w), \pi_1(b|w)) = (0, 1, 0, 1)$. By Bayes Rule the assessment for player 2 at the second information set has to be

$$\alpha_2(s|b) = \frac{\pi_1(b|s)\Pr(s)}{\pi_1(b|s)\Pr(s) + \pi_1(b|w)\Pr(w)} = 1 - \alpha_2(w|b) \tag{11}$$

$$\alpha_2(s|b) = \frac{1 \times 0.9}{1 \times 0.9 + 1 \times 0.1} = 0.9 = 1 - \alpha_2(w|b) \tag{12}$$

$\alpha_2 = (\alpha_2(s|q), \alpha_2(w|q), \alpha_2(s|b), \alpha_2(w|b)) = (x, 1-x, 0.9, 0.1)$ with x still undetermined. As before, we can construct an assessment such that $x \leq \frac{1}{2}$. Then, $\pi_2 = (\pi_2(n|q), \pi_2(d|q), \pi_2(n|b), \pi_2(d|b)) = (0, 1, 1, 0)$ will be sequentially rational.

Finally, we have to check if player 1 wants to play π_1 when his expectations are that $\mu_1 = \pi_2 = (0, 1, 1, 0)$. The answer is yes. By deviating he will get beaten up again.