Notes on Extensive form games
Course Econ 201B

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January 21, 2005

1 Introduction

The extensive form of a game conveys more information about the game. In a sense, the normal form of a game is a static concept since it does not explicitly model in what order the game is played. The extensive form of the game is dynamic in the sense that it captures the order of play and features related to the timing of what we are modeling.

The extensive form allows for an equilibrium concept that refines Nash equilibrium. This concept is called subgame perfection and can intuitively be thought of as eliminating threats that are not credible.

2 Subgame Perfection

2.1 What is a subgame?

A subgame is every subset of the game tree that looks like a game. This means that it has an initial node, contains all nodes that follow this initial node; in fact, it contains only those nodes that can be traced to the initial node. Additionally, a subgame can only start at information sets that contain a single node and information sets cannot be divided to form a subgame. Since the original game is also a subgame according to this definition, subgames that are not the whole game are sometimes called "proper subgames".

2.2 Definition of subgame perfection

Definition 1 *A subgame perfect equilibrium is a profile of strategies that are a Nash Equilibrium in every subgame (including the game as a whole).*

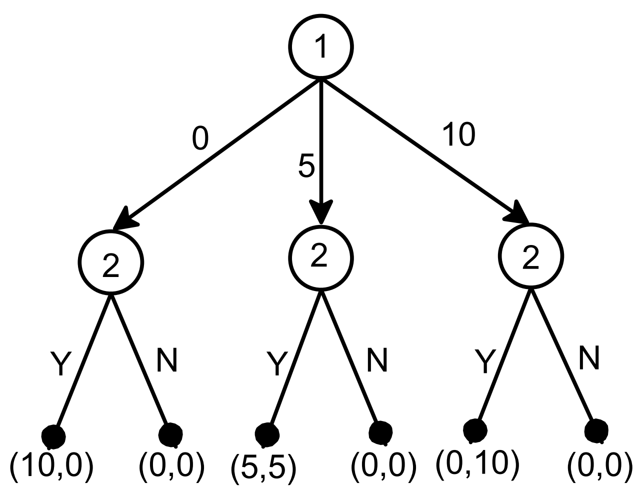
Note: since we also include the game as a whole, subgame perfection is a refinement of the concept of Nash equilibrium.

3 Examples

3.1 A simple ultimatum bargaining game

In the original ultimatum bargaining game a player has to make an offer on how to share 10 dollars in pennies and the other player can accept of reject it. In this game the player making the offer has 1000 strategies and the player receiving the offer has 2^{1000} strategies. Let's consider a simpler case in which the proposing player can only use 5 dollar bills. Then, only \$0, \$5 or \$10 are possible offers. In this modified game player 1 has 3 different strategies and player 2 has $2^3 = 8$ strategies.

3.1.1 The extensive form of the game



3.1.2 The Normal form of the game

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
\$0	**10,0**	**10,0**	**10,0**	**10,0**	0,**0**	0,**0**	**0,0**	**0,0**
\$5	5,**5**	5,**5**	0,0	0,0	**5,5**	**5,5**	**0**,0	**0**,0
\$10	0,**10**	0,0	0,**10**	0,0	0,**10**	0,0	**0,10**	**0**,0

There are several Nash equilibria. In fact all possible outcomes occur in at least one equilibrium. What happens if we apply subgame perfection? Player 2 would never reject anything, except when he is offered \$0. In that case he is indifferent.

The only pure strategy Nash equilibria that survive the refinement of subgame perfection are $(\$0, yyy)$ and $(\$5, nyy)$. In fact, if we modify the game a little bit, so that the lowest that can be offered is ε we end up with just one outcome prediction:

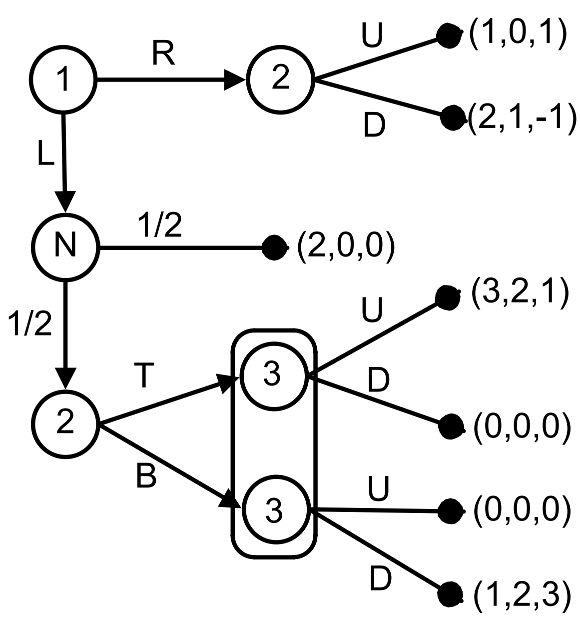
Normal form of the modified game:

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
$\$\varepsilon$	**10-**ε**,**ε	**10-**ε**,**ε	**10-**ε**,**ε	**10-**ε**,**ε	0,0	0,0	0,0	0,0
\$5	5,**5**	5,**5**	0,0	0,0	**5,5**	**5,5**	**0**,0	**0**,0
\$10	0,**10**	0,0	0,**10**	0,0	0,**10**	0,0	**0,10**	**0**,0

Now player 2 is not indifferent between accepting or rejecting the lowest offer. He will accept anything. This implies that there is a unique subgame perfect Nash equilibrium: $(\$\varepsilon, yyy)$.

3.2 Another example

3.2.1 The extensive form



3.2.2 Construct the normal form

Player 1 has two strategies: L and R

Player 2 has four strategies: TU, TD, BU, BD

Player 3 has two strategies: U, D

In the following tables player 2 is the row player and player 3 the column player

Player 1 plays L

	U	D
TU	2.5,1,0.5	1,0,0
TD	2.5,1,0.5	1,0,0
BU	1,0,0	1.5,1,1.5
BD	1,0,0	1.5,1,1.5

Player 1 plays R

	U	D
TU	1,0,1	1,0,1
TD	2,1,-1	2,1,-1
BU	1,0,1	1,0,1
BD	2,1,-1	2,1,-1

Is any strategy strictly dominated?

Are there any payoff equivalent strategies that can be eliminated to get a smaller reduced normal form?

3.2.3 Find all pure strategy Nash equilibria

Best responses

Player 1 plays L

	U	D
TU	**2.5**,**1**,**0.5**	**1**,0,0
TD	**2.5**,**1**,**0.5**	1,0,0
BU	**1**,0,0	**1.5**,**1**,**1.5**
BD	1,0,0	1.5,**1**,**1.5**

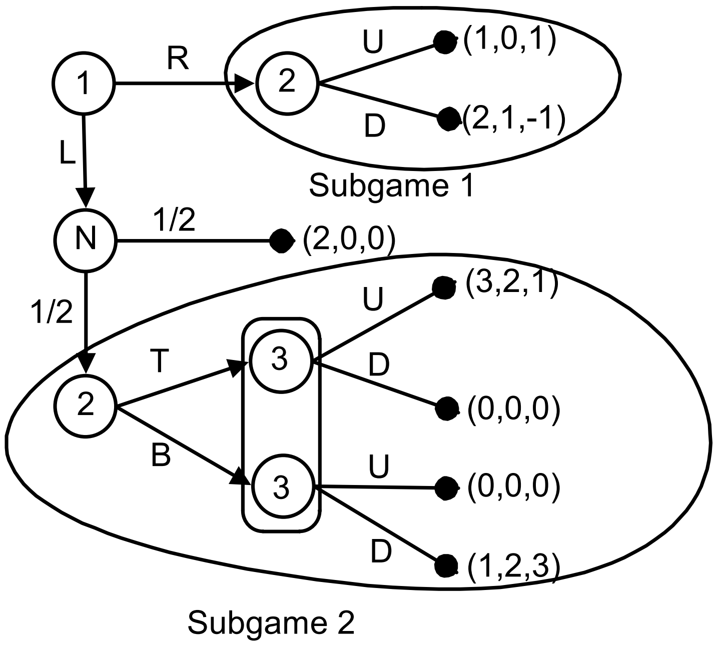
Player 1 plays R

	U	D
TU	1,0,**1**	**1**,0,**1**
TD	2,**1**,**-1**	**2**,**1**,**-1**
BU	**1**,0,**1**	1,0,**1**
BD	**2**,**1**,**-1**	**2**,**1**,**-1**

There are six pure strategy Nash equilibria: (L, TU, U), (L, TD, U), (L, BU, D), (R, TD, D), (R, BD, U), (R, BD, D)

3.2.4 Which ones are subgame perfect NE?

There are only two proper subgames in this game where players have the initial move (there is also one that starts at the node where nature has the move). We will call them Subgame 1 and Subgame 2



Start by analyzing the first subgame. It is optimal for player 2 to choose D, yielding a payoff of $(2, 1, -1)$ to all players.

Now let's proceed with the second subgame. Its normal form is given in the following table. We can disregard payoffs to player 1 since he does not participate in this subgame.

	U	D
T	**2,1**	0,0
B	0,0	**2,3**

This subgame has two pure Nash equilibria: (T, U) and (B, D) which yield payoffs $(3, 2, 1)$ and $(1, 2, 3)$ respectively. There is also a mixed equilibrium of the form $(\sigma_2 (T), \sigma_2 (B), \sigma_3 (U), \sigma_3 (D)) = \left(\frac{3}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right)$, yielding a payoff of $\left(\frac{5}{4}, 1, \frac{3}{4}\right)$.

Now consider the subgame starting at the node where Nature has the move. In this subgame there is not much to analyze since no interesting action takes place. With probability $\frac{1}{2}$ we reach the payoff $(2, 0, 0)$ and with probability $\frac{1}{2}$

we reach Subgame 2, which we already analyzed. The only thing we need to do is to adjust the payoffs arising from the Nash equilibria we found for Subgame 2 by using the formula for expected utility. The payoffs for the three NE we found before are $(2.5, 1, 0.5)$, $(1.5, 1, 1.5)$ for the pure NE and $\left(\frac{13}{8}, 0.5, \frac{3}{8}\right)$ for the mixed equilibrium.

The problem is now reduced to a decision problem for player 1. By choosing R he chooses the payoff vector $(2, 1, -1)$ by choosing L he may arrive at three different payoff profiles: $(2.5, 1, 0.5)$, $(1.5, 1, 1.5)$ and $\left(\frac{13}{8}, 0.5, \frac{3}{8}\right)$. Clearly, player 1 will only choose L in the case where players 2 and 3 play (T, U) in subgame 2 and R otherwise.

Therefore, the pure subgame perfect NE are: (L, TD, U) and (R, BD, D)

In the mixed subgame perfect NE player 1 plays $(\sigma_1(L), \sigma_1(R)) = (0, 1)$, player 2 plays $(\sigma_2(TU), \sigma_2(TD), \sigma_2(BU), \sigma_2(BD)) = \left(0, \frac{3}{4}, 0, \frac{1}{4}\right)$ and player 3 plays $(\sigma_3(U), \sigma_3(D)) = \left(\frac{1}{2}, \frac{1}{2}\right)$.