The Folk Theorem with Imperfect Public Information

The Stage Game

players $i = 1,\ldots,n$

pure actions $a_i \in A_i$ which has m_i elements

public outcomes $y \in Y$ which has m elements

common knowledge probabilities $\pi_y(a)$

normal form $g_i(a)$

mixed actions α

some obvious notation $g_i(\alpha)$

also write $\alpha = (\alpha_i, \alpha_{-i})$ a lot

The Repeated Game

players maximize average present value at common δ

public history; private history; strategies

Perfect Public Equibrium

a Nash equilibrium in which only public histories matter and forming another Nash equilibrium after every public history

Objective

characterize the set of long-run players payoffs

$E(\delta) \subset \Re^n$

as $\delta \to 1$

Basics

$V \subset \Re^n$ convex combinations of payoffs to long-run players

well known results: $E(\delta) \subset V$ are compact

$$E \equiv \lim_{\delta \to 1} E(\delta)$$

Enforceability (Abreu-Pearce-Stachetti)

$\delta, W \subseteq \Re^L, v \in \Re^n, \alpha \; w : Y \to W$

satisfy

$$v_i = (\geq)(1-\delta) g_i(a_i, \alpha_{-i}) + \delta \sum_{y \in Y} \pi_y(a_i, \alpha_{-i}) w_i(y)$$

$$\text{if } \alpha_i(a_i) > 0 \quad (\alpha_i(a_i) = 0)$$

We say

α is enforceable

w enforces

v is generated

Generation and Equilibrium

if W is contained in the set it generates, we say it is *self generated*

compact self-generated sets consist of equilibrium payoffs; the set of all equilibrium payoffs is the maximal self-generated set

(for any discount factor)

the set of equilibrium payoffs is closed, but may fail to be convex, and may generally be quite nasty

(unless we assume public randomization which we did not)

Patience and Half-spaces

Key idea: as long-run players are increasingly patient $E(\delta)$ becomes increasingly convex. A convex set can be represented as a union of tangent halfspaces.

Define $H(k,\lambda) \equiv \{v \in \Re^L | \lambda \cdot v \leq k\}$ to be the half-space in direction λ of size k.

The key is to study what payoff vectors can be generated by half-spaces.

Scores

maximal score by α in direction λ

$$k^*(\alpha, \lambda, \delta) \equiv \max \lambda \cdot v$$

$$\alpha, v \text{ enforceable on } H(\lambda, \lambda \cdot v)$$

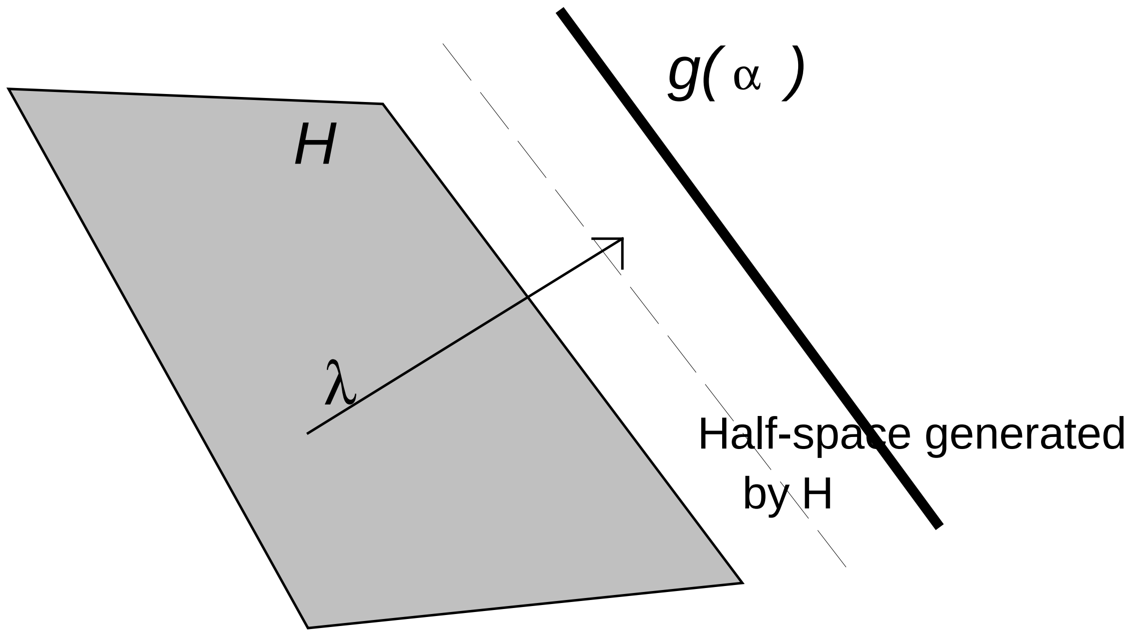
notice that this is a perfectly ordinary finite dimensional LP problem (since *H* is defined by a linear inequality)

Lemma 3.1: $k^*(\alpha, \lambda, \delta) = k^*(\alpha, \lambda) \leq \lambda \cdot g(\alpha)$

Can't get outside the socially feasible set

Generation on Half Spaces

Note that the set of points generated by a half-space is itself a half-space pointing in the same direction



Combining Half-Spaces

maximal score in a direction $k^*(\lambda) \equiv \sup_\alpha k^*(\alpha, \lambda)$

$H^*(\lambda)$ the corresponding half-space

$$Q \equiv \bigcap_\lambda H^*(\lambda)$$

Easy result: $E(\delta) \subseteq Q$

Hard result: $\dim Q = n \Rightarrow E \supseteq Q$

Why the Hard Result is True

Look at smooth approximation interior to Q

Discount factor close to one looks at things through magnifying glass

Operate locally (local generation)

Paste together by compactness

Picture

Failure of Dimensionality

Remark: If *Q* has too low dimension, the entire argument may be repeated by recalculating *Q* subject to the additional restriction of lower dimension. Eventually (since n is finite) we find *E*.

Typical application: use some special structure to reduce the number of profiles and directions that must be examined to compute *Q.* For example, in some games it can be shown that only pure actions matter; generally certain directions are much harder than others.

Regular, Horizontal and Vertical Half-spaces

Regular half-spaces: informational condition

Top half-spaces: efficiency

Bottom half-spaces: need enforceability of minmax, or else do Nash threats

The Matrix

For an individual

$$\Pi_i(\alpha) = (\pi_y(a_i, \alpha_{-i}))\, |_{y \in Y}^{a_i \in A_i}$$

could have full rank if $\# Y \geq \# A_i$

For a pair

$$\Pi_{ij}(\alpha) = \begin{bmatrix} (\pi_y(a_i, \alpha_{-i}))\, |_{y \in Y}^{a_i \in A_i} \\ (\pi_y(a_j, \alpha_{-j}))\, |_{y \in Y}^{a_j \in A_j} \end{bmatrix}$$

cannot have full rank, has at least one linear dependency as

a linear combination of each of the two matrices must equal $\pi_y(\alpha)$

Information Conditions: At a Point

- enforcible
- pairwise identifiability
- b.r. for player *i*
- coordinate vs. regular hyperplanes

- enforcible + b.r. => coordinate
- enforcible + pairwise identifiability => regular

- full rank => enforcible
- pairwise full rank => pairwise identifiability

Information Conditions: Global

- pure pareto efficient is pairwise identifiable => Nash threat
- all pairs exists pairwise full rank => full folk