Collusion, Randomization and Leadership in Groups

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Overview

- players are exogenously partitioned into groups within which players are symmetric
- given the play of the other groups there may be several symmetric equilibria for a particular group
- if group can collude they will agree to choose the equilibrium most favorable for its members
- this leads to non-existence
- augment the definition with shadow mixing
- show the limit of games with perturbed beliefs
- show equivalent to a leadership game
- builds on models used in mechanism design theory to study collusion in auctions

A Motivating Example

three players

first two players form a collusive group and the third acts independently

theory: given the play of player 3, players 1 and 2 should agree on the incentive compatible pair of (mixed) actions that give them the most utility

each player chooses one of two actions, C or D and payoffs given in bi-matrix form

Payoffs

player 3 plays C payoff matrix for the actions of players 1 and 2 is a symmetric Prisoner's Dilemma game in which player 3 prefers that 1 and 2 cooperate C

	C	D
C	$6, 6, 5$	$0, 8, 5$
D	$8, 0, 5$	$2, 2, 0$

If player 3 plays D the payoff matrix for the actions of players 1 and 2 is a symmetric coordination game in which player 3 prefers that 1 and 2 defect D

	C	D
C	$6, 6, 0$	$4, 4, 0$
D	$4, 4, 0$	$5, 5, 5$

Equilibrium

α^i probability with which player i plays C

set of equilibria for players 1 and 2 given α^3

$\alpha^3 > 1/2$ then D strictly dominant for both player 1 and 2 so they play D,D

$\alpha^3 = 1/2$ two equilibria, both symmetric at C,C and D,D

$\alpha^3 < 1/2$ three equilibria, all symmetric, C,C, D,C and a strictly mixed equilibrium $\alpha^1 = \alpha^2 = (1/3)(1+\alpha^3)/(1-\alpha^3)$

Optimal Collusion

$\alpha^3 > 1/2$ no choice, they have to do D,D (remark: also the unique correlated equilibrium)

$\alpha^3 \leq 1/2$ get 6 at C,C equilibrium and strictly less than 6 at any other correlated strategy

no ambiguity about the preferences of the group: they unanimously agree in each case as to which is the best equilibrium.

group best response

$\alpha^3 > 1/2$ play D,D

$\alpha^3 \leq 1/2$ play C,C

best response of 3

group at D,D play D so $\alpha^3 = 0$ at C,C $\alpha^3 = 1$

no equilibrium

Does this make sense?

a small change in the probability of α^3 leads to an abrupt change in the behavior of the group

but how can the group know α^3 so exactly?

rather it makes sense that as the beliefs of a group change the probability with which they play different equilibria varies continuously

$\alpha^3 = 0.499$ versus $\alpha^3 = 0.501$

the theory: player 1 and 2 with probability 1 agree that $\alpha^3 \leq 0.5$ in the former case and in the latter case that $\alpha^3 > 0.5$

perhaps it makes more sense to say that they agree that $\alpha^3 \leq 0.5$with 90% of the time in the former case and mistakenly agree that $\alpha^3 > 0.5$ 10% of the time?

The Cheshire Cat

for the moment suppose that in that limit only the randomization will remain

assume that randomization is possible at the critical point

when $\alpha^3 = 0.5$ and the incentive constraint exactly binds, the equilibrium “assigns” an arbitrary probability to C,C being the equilibrium

if we have 0.5 chance of C,C and D,D then 3 is indifferent and we have an equilibrium

The Exogenous Group Model

players $i = 1 \dots I$ and groups $k = 1, \dots K$

actions available for members of group k are A^k a finite set

a fixed assignment of players to groups $k(i)$

all players within a group are symmetric; utility of player i is $u^{k(i)}(a^i, a^{-i})$ and invariant with respect to within group permutations of the labels of other players

$\mathcal{A}^k$ are mixed actions for a member of group k, profiles of play chosen from this set represent the universe in which in-group equilibria reside

each group is assumed to possess a private randomizing device observed only by members of that group that can be used to coordinate group play

restrict to finite subset $A^{kR} \subseteq \mathcal{A}^k$ and consider only in-group equilibria for group k in which all players choose the same action $a^k \in A^{kR}$

Discussion

finiteness simplifies probability distributions over a continuous set

it creates a complication because in-group equilibria may not exist in a finite set

will use approximate equilibrium to take care of that

now write $u^{k(i)}(a^i, a^{k(i)}, \alpha^{-k(i)})$

Collusion

groups collude but must respect incentive constraints

group objective: maximize the common utility that they receive when all are treated equally

Incentive Slack

strictly positive numbers $v^k > 0$ measuring in utility units the violation of incentive constraints that are allowed

gain function

$$G^k(a^k, \alpha^{-k}) = \max_{a^i \in A^k \cup \{a_0^k\}}[u^k(a^i, a^k, \alpha^{-k}) - u^k(a^k, a^k, \alpha^{-k})]$$

degree to which incentive constraint is violated

gain strictly less than v^k then a^k must be chosen by the group if it is to the benefit of the group to do so

gain is greater than v^k then the group cannot choose a^k

gain is exactly v^k group may mix with any probability onto a^k if it is to their benefit to do so

Collusion Constrained Equilibrium

$U^k(\alpha^{-k}) = \max_{\{a^k | G^k(a^k,\alpha^{-k}) < v^k\}} u^k(a^k, a^k, \alpha^{-k})$

most utility attainable against α^{-k} when the incentive constraints are violated by strictly less than v^k

$B^k(\alpha^{-k}) = \{a^k | G^k(a^k, \alpha^{-k}) \leq v^k, u^k(a^k, a^k, \alpha^{-k}) \geq U^k(\alpha^{-k})\}$

feasible group actions for α^{-k} - the *shadow response set*

contrast with

$$\overline{B}^k(\alpha^{-k}) = \arg\max_{\{a^k | G^k(a^k,\alpha^{-k}) \leq v^k\}} u^k(a^k, a^k, \alpha^{-k}) \subseteq B^k(\alpha^{-k})$$

A *collusion constrained equilibrium* is an α^k for each group that places weight only on $B^k(\alpha^{-k})$.

Incentive Compatible Games

If A^{kR} contains a relatively fine grid of mixtures there will be an ϵ-Nash equilibrium with a small value of ϵ

v^k strictly bigger than ϵ the group can find an action that is guaranteed to satisfy the incentive constraints to the required degree

$g^k = \max_{\alpha^{-k}} \min_{a^k \in A^{kR}} G^k(a^k, \alpha^{-k})$: regardless of the behavior of the other groups there is always a g^k approximate equilibrium within the group.

A game is *incentive compatible* if $v^k > g^k$ for all k

A Basic Continuity Result

Lemma: (i) In an incentive compatible game $\overline{B}^k(\alpha^{-k})$ is non-empty for all α^{-k}; (ii) every α^{-k} has an open neighborhood $\mathcal{A}$ such that $\tilde{\alpha}^{-k} \in \mathcal{A}$ implies that $B^k(\tilde{\alpha}^{-k}) \subseteq B^k(\alpha^{-k})$

implies existence

Random Beliefs

given the true play $\alpha^{-k}$of the other groups, there is a common belief $\tilde{\alpha}^{-k}$ by group k that is a random function of that true play

An ϵ*-random group belief model* is a density function $f^k(\tilde{\alpha}^{-k}|\alpha^{-k})$ that is a continuous as a function of $\tilde{\alpha}^{-k}, \alpha^{-k}$ and satisfies $\int_{|\tilde{\alpha}^{-k}-\alpha^{-k}|\leq\epsilon} f_\epsilon^k(\tilde{\alpha}^{-k}|\alpha^{-k})d\tilde{\alpha}^{-k} \geq 1-\epsilon$.

Example of a Random Belief Model

M^k be the number of actions in A^k

$h(\epsilon) = (\epsilon/2)^2 M^k / (M^k - (\epsilon/2)^2)$

fix a strictly positive probability vector over A^{-k} denoted by $\hat{\alpha}^{-k}$ and call ϵ-*Dirichlet belief model* the Dirichlet distribution with parameters

$\frac{1}{h(\epsilon)}(1 - \frac{\epsilon}{2\sqrt{2}})\alpha^{-k}(a^{-k}) + \frac{\epsilon}{2\sqrt{2}}\widehat{\alpha}^{-k}(a^{-k}).$

Random Belief Equilibrium

$F^k(\alpha^{-k})$ be any probability distribution over $\overline{B}^k(\alpha^{-k})$ measurable as a function of α^{-k}.

$R^k(a^k|\alpha^{-k}) = \int F^k(\tilde{\alpha}^{-k})[a^k] f^k(\tilde{\alpha}^{-k}|\alpha^{-k}) d\tilde{\alpha}^{-k}$.

an ϵ*-random belief equilibrium* as an α_ϵ such that $\alpha_\epsilon^k = R^k(\alpha_\epsilon^{-k})$.

Theorem: *Fix a family of* ϵ*-random group belief models, an* $F^k(\alpha^{-k})$ *and an incentive compatible game. Then for all* $\epsilon > 0$ *there exist* ϵ*-random group equilibria.Further, if* α_ϵ *are* ϵ*-random belief equilibria and* $\lim_{\epsilon \to 0} \alpha_\epsilon = \alpha$ *then* α *is a collusion constrained equilibrium.*

What Difference Do Collusion Constraints Make?

3C		C	D	3D		C	D
	C	$6, 6, 5$	$0, 8, 5$		C	$6, 6, 0$	$4, 4, 0$
	D	$8, 0, 5$	$2, 2, 0$		D	$4, 4, 0$	$5, 5, 5$

independent players

unique Nash equilibrium DDD (5,5,5)

group ignores incentive constraints

unique outcome CCC (6,6,5)

collusion constrained

group shadow mixes 50-50 CC and 3 mixes 50-50 (4.75,4.75,2.5)

mechanism designer with safe alternative of (4.9,4.9,4.9)

Leadership Equilibrium

group leaders serve as explicit coordinating devices for groups

we do not want leaders to issue instructions that members would not wish to follow

give them incentives to issue instructions that are incentive compatible by allowing group members “punish'” their leader

here v^k has a concrete interpretation as the leader's valence: the higher v^k the more members are ready to give up to follow the leader.

A non-cooperative game of leaders

Each group is represented by two virtual players: leader and *evaluator* with the same underlying preferences as the group members

Each leader has a punishment utility $\underline{u}^k < \min_{a^j, a^k, a^{-k}} u^k(a^j, a^k, a^{-k})$. The game goes as follows:

Stage 1: each leader privately chooses an action plan $a^k \in A^{kR}$: conceptually these are orders given to the members who must obey the orders.

Stage 2: the evaluator observes the action plan of the leader of his own group

Stage 3: the evaluator chooses a response a^i

Payoffs: if the evaluator chooses a^k he receives utility $u^k(a^k, a^k, a^{-k}) + v^k$; if he chooses $a^i \neq a^k$ he receives utility $u^k(a^i, a^k, a^{-k})$. If the evaluator chooses $a^i \neq a^k$ the leader is deposed and gets utility $\underline{u}^k$. Otherwise the leader gets utility $u^k(a^k, a^k, a^{-k})$

Equivalence of Leadership Equilibria

Note that the leader and evaluator do not learn what the other groups did until the game is over.

Theorem: *In an incentive compatible game* α *are sequential equilibrium choices by the leaders if and only if* $\alpha^k(a^k) > 0$ *implies* $a^k \in B^k(\alpha^{-k})$