Bayesian Games and Mechanism Design

Definition of Bayes Equilibrium

Harsanyi [1967]

- What happens when players do not know one another's payoffs?
- Games of "incomplete information" versus games of "imperfect information"
- Harsanyi's notion of "types" encapsulating "private information"
- Nature moves first and assigns each player a type; player's know their own types but not their opponents' types
- Players do have a common prior belief about opponents' types

Bayesian Games

There are a finite number of types $\theta_i \in \Theta_i$

There is a common prior $p(\theta)$ shared by all players

$p(\theta_{-i} \mid \theta_i)$ is the conditional probability a player places on opponents' types given his own type

The *stage* game has finite action spaces $a_i \in A_i$ and has utility functions $u^i(a,\theta)$

Bayesian Equilibrium

A *Bayesian Equilibrium* is a Nash equilibrium of the game in which the strategies are maps from types $s_i : \Theta_i \to A_i$ to stage game actions A_i

This is equivalent to each player having a strategy as a function of his type $s_i(\theta_i)$ that maximizes conditional on his own type θ_i (for each type that has positive probability)

$$\max_{s_i} \sum_{\theta_{-i}} u_i(s_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) p(\theta_{-i} \mid \theta_i)$$

Cournot Model with Types

- A duopoly with demand given by $p = 17 - x$
- A firm's type is its cost, known only to that firm: each firm has a 50-50 chance of cost constant marginal cost 1 or 3.

profits of a representative firm

$$\pi_i(c_i, x) = [17 - c_i - (x_i + x_{-i})]x_i$$

Let us look for the symmetric pure strategy equilibrium

Finding the Bayes-Nash Equilibrium

x^1, x^3 will be the output chosen in response to cost

$$\pi_i(x_i, c_i) = .5\big[17 - c_i - (x_i + x^1)\big]x_i$$
$$+.5\big[17 - c_i - (x_i + x^3)\big]x_i$$

maximize with respect to x_i and solve to find

$x^1 = 11/2$, $x^3 = 9/2$

industry output

probability ¼ 11

probability ½ 10

probability ¼ 9

Suppose by contrast costs are known

If both costs are 1 then competitive output is 16 and Cournot output is 2/3rds this amount 10 2/3

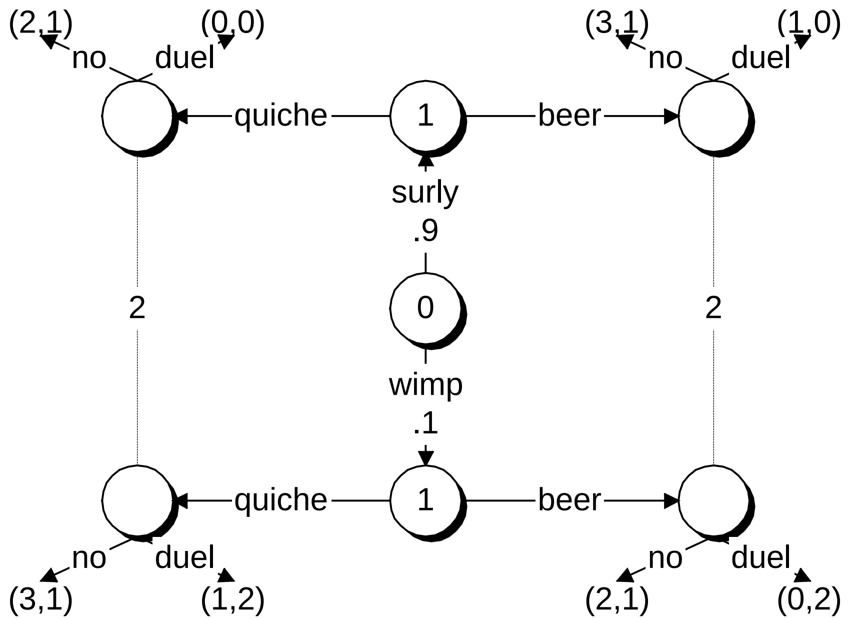
If both costs are 3 then competitive output is 14 and Cournot output is 9 1/3

If one cost is 1 and one cost is 3 Cournot output is 10

With known costs, mean industry output is the same as with private costs, but there is less variation in output

Sequentiality and Signaling

Cho-Kreps [1987]



Sequentiality

Kreps-Wilson [1982]

Subforms

Beliefs: *assessment* a_i for player *i* probability distribution over nodes at each of his information sets; *belief* for player *i* is a pair $b_i \equiv (a_i, \pi^i_{-i})$, consisting of *i*'s assessment over nodes a_i, and *i*'s expectations of opponents' strategies $\pi^i_{-i} = (\pi^i_j)_{j \neq i}$

Beliefs come from strictly positive perturbations of strategies

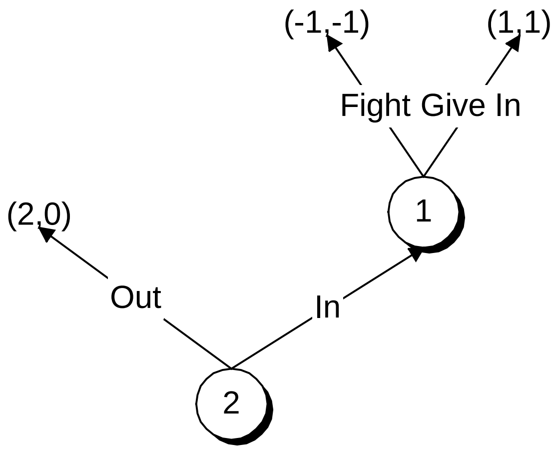
belief $b_i \equiv (a_i, \pi^i_{-i})$ is *consistent* (Kreps and Wilson [17]) if $a_i = \lim_{n\to\infty} a_i^n$ where a_i^n obtained using Bayes rule on a sequence of strictly positive strategy profiles of the opponents, $\pi^{i,m}_{-i} \to \pi_{-i}$

given beliefs we have a well-defined decision problem at each information set; can define optimality at each information set

A sequential equilibrium is a behavior strategy profile π and an assessment a_i for each player such that (a_i, π^i_{-i}) is consistent and each player optimizes at each information set

Chain Store Paradox

Kreps-Wilson [1982], Milgrom-Roberts [1982]



finitely repeated model with long-run versus short-run

Reputational Model

two types of long-run player $\omega \in \Omega$

“rational type” and “committed type”

“committed type” will fight no matter what

types are privately known to long-run player, not known to short run player

Kreps-Wilson; Milgrom-Roberts

Solve for the sequential equilibrium; show that at the time-horizon grows long we get no entry until near the end of the game

“triumph of sequentiality”

The Holdup Problem

- Chari-Jones, the pollution problem
- problem of too many small monopolies

ρ is the profit generated by an invention with a monopoly with a patent, drawn from a uniform distribution on $[0,1]$, private to the inventor

ϕ^F is the fraction of this profit that can be earned without a patent

To create the invention requires as input N other existing inventions

It costs ε / N to make copies of these other inventions, where $\varepsilon < 1/2$ and $\varepsilon / \phi^F < 1$

Case 1: Competition

if $\phi^F \rho \geq \varepsilon$ the new invention is created, probability is $1 - \varepsilon / \phi^F$.

Case 2: Patent

Each owner of the existing inventions must decide a price p_i at which to license their invention; ϕN current inventions are still under patent

Subgame Perfection/Sequentiality implies that the new invention is created when $(\phi + \phi^F)\rho \geq \sum_i p_i + \varepsilon$

Profit of preexisting owners $(1 - \frac{(\phi N - 1)p + p_i - \varepsilon}{\phi + \phi^F})p_i$

FOC $1 - \frac{(\phi N - 1)p + 2p_i + \varepsilon}{\phi + \phi^F} = 0$

symmetric equilibrium $p = (\phi + \phi^F - \varepsilon)/(\phi N + 1)$; $\sum_i p_i = \phi N p$

corresponding probability of invention $(\phi + \phi^F - \varepsilon)/[(\phi + \phi^F)(\phi N + 1)]$

Micro Mechanism Design

An "auction" problem

- Single seller has a single item
- Seller does not value item
- Two buyers with independent valuations

$0 \leq v^l < v^h$ low and high valuations

$\pi^l + \pi^h = 1$ probabilities of low and high valuations what is the best way to sell the object

- Auction
- Fixed price
- Other

The Revelation Principle

Design a game for the buyers to play

- Auction game
- Poker game
- Etc.

Design the game so that there is a Nash equilibrium that yields highest possible revenue to the seller

The revelation principle says that it is enough to consider a special game

- strategies are “announcements” of types
- the game has a “truthful revelation” equilibrium

In the Auction Environment

Fudenberg and Tirole section 7.1.2

q^l, q^h probability of getting item when low and high

p^h, p^l expected payment when low and high *individual rationality constraint*

(IR) $q^i v^i - p^i \geq 0$

- if you announce truthfully, you get at least the utility from not playing the game

incentive compatibility constraint

(IC) $q^i v^i - p^i \geq q^{-i} v^i - p^{-i}$

- you gain no benefit from lying about your type

the incentive compatibility constraint is the key to equilibrium

Other constraints

q^l, q^h probability of getting item when low and high

they can't be anything at all:

probability constraints

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i / 2$

(win against other type, 50% chance of winning against self)

(2) $\pi^l q^l + \pi^h q^h \leq 1/2$

(probability of getting the good before knowing type less than 50%)

Seller Problem

Maximize seller utility $U = \pi^l p^l + \pi^h p^h$

Subject to IC and IR

To solve the problem we make a guess:

IR binds for low value

$q^l v^l - p^l = 0$

IC binds for high value

$q^h v^h - p^h = q^l v^h - p^l$

The solution

$p^l = q^l v^l$ from low IR

substitute into high IC

$$p^h = (q^h - q^l)v^h + q^l v^l$$

plug into utility of seller

$$U = \pi^l q^l v^l + \pi^h \left((q^h - q^l)v^h + q^l v^l \right)$$

$$U = q^l(\pi^l v^l - \pi^h v^h + \pi^h v^l) + \pi^h q^h v^h$$

$\pi^l + \pi^h = 1$ so

$$U = q^l(v^l - \pi^h v^h) + \pi^h q^h v^h$$

Case 1: $v^l > \pi^h v^h$

$$U = q^l(v^l - \pi^h v^h) + \pi^h q^h v^h$$

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i / 2$

(2) $\pi^l q^l + \pi^h q^h \leq 1/2$

Make q^l, q^h large as possible so

$\pi^l q^l + \pi^h q^h = 1/2$

$$U = \frac{1/2 - \pi^h q^h}{\pi^l}(v^l - \pi^h v^h) + \pi^h q^h v^h$$

$$U = \frac{1}{2\pi^l}(v^l - \pi^h v^h) + q^h \frac{\pi^h}{\pi^l}(v^h - v^l)$$

so q^h should be as large as possible

$$q^h = \pi^l + \pi^h/2$$

plug back into (2) to find

$$q^l = \pi^l/2$$

expected payments

$$p^l = q^l v^l,\; p^h = (q^h - q^l)v^h + q^l v^l$$

$$p^l = v^l \pi^l/2$$

$$p^h = v^h/2 + \pi^l v^l/2$$

Implementation of Case 1

modified auction: each player announces their value

the highest announced value wins

if there is a tie, flip a coin

if the low value wins, he pays his value

if the high value wins he pays

$$\frac{p^h}{q^h} = \frac{v^h/2 + \pi^l v^l/2}{\pi^l + \pi^h/2}$$

under these rules

probability that high type wins is $q^h = \pi^l + \pi^h/2$

probability that low type wins is $q^l = \pi^l/2$

just as in the optimal mechanism

this means the expected payments are the same too

Case 2: $v^l < \pi^h v^h$

$U = q^l(v^l - \pi^h v^h) + \pi^h q^h v^h$

(1) $0 \leq q^i \leq \pi^{-i} + \pi^i/2$

(2) $\pi^l q^l + \pi^h q^h \leq 1/2$

Make q^h large as possible, q^l as small as possible

$q^h = \pi^l + \pi^h/2$

$q^l = 0$

expected payments

$p^l = q^l v^l$, $p^h = (q^h - q^l)v^h + q^l v^l$

$p^l = 0$

$p^h = (\pi^l + \pi^h / 2)v^h$

Implementation of Case 2

set a fixed price equal to the highest valuation

$$v^h = \frac{p^h}{q^h} = \frac{(\pi^l + \pi^h/2)v^h}{\pi^l + \pi^h/2}$$