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Debt constraints and equilibrium in infinite horizon economies with incomplete markets

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Abstract

This paper defines a notion of an equilibrium and a pseudo-equilibrium for infinite horizon economies with incomplete asset markets. This definition generalizes the usual ones for finite horizon economies with incomplete markets and for infinite horizon economies with complete markets. We establish the existence of a pseudo-equilibrium when assets are short-lived and denominated in general commodity bundles; we obtain a true equilibrium when assets are denominated solely in a single numeraire commodity, or in units of account. It seems to us that the notion of an equilibrium we define is a natural and compelling one; as evidence, we show that our notion actually coincides with several other – apparently quite distinct – notions of an equilibrium.

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1. Introduction

This paper defines a notion of an equilibrium and a pseudo-equilibrium for infinite horizon economies with incomplete asset markets. This definition general-

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izes the usual ones for finite horizon economies with incomplete markets and for infinite horizon economies with complete markets. We establish the existence of a pseudo-equilibrium when assets are short-lived and denominated in general commodity bundles; we obtain a true equilibrium when assets are denominated solely in a single numeraire commodity, or in units of account. It seems to us that the notion of an equilibrium we define is a natural and compelling one; as evidence, we show that our notion actually coincides with several other – apparently quite distinct – notions of an equilibrium.

The crucial issue that divides the infinite horizon setting from the finite horizon setting is the nature of debt constraints. In the finite horizon setting the constraint that there be no debt following the terminal date, together with the budget constraint, imply limits on debt at earlier dates. In the infinite horizon setting these terminal debt constraints – and the implied debt constraints at earlier dates – are absent. If no additional debt constraints were imposed, then an equilibrium could not possibly exist: all traders would attempt to finance unbounded levels of consumption by unbounded levels of borrowing. When markets are complete, such Ponzi schemes may be ruled out by the simple requirement that debt at each date/event never exceeds the current value of future endowments; this is frequently called a *solvency requirement*. Completeness of markets guarantees that solvency is an unambiguous requirement. Moreover, in the presence of appropriate assumptions about preferences and endowments, it is sufficient to guarantee that an equilibrium exists (see Bewley 1972, for instance). However, when markets are incomplete, solvency is no longer an unambiguous requirement. When markets are incomplete, marginal rates of substitution for different traders may not be equal at equilibrium; as a consequence, traders may not agree on current value prices. [1]

In the complete markets setting, an alternative formulation of the solvency requirement is that, at each node, *almost* all the debt can be repaid in finite time; this latter formulation has the advantage that it makes perfect sense in the incomplete markets setting as well. [2] We say that such debt constraints are *finitely effective*. This condition expresses the same intuition as the usual solvency condition and rules out Ponzi schemes. We show that it also meets the basic consistency test of sufficing for the existence of what we term a *finitely effective equilibrium* (with appropriate assumptions on preferences and endowments). A finitely effective equilibrium reduces to the usual notions of equilibrium in the infinite horizon setting with complete markets and in the finite horizon setting with incomplete markets.

[1] Magill and Quinzii (1992) construct a theory of debt constraints in which the debts of different traders are evaluated according to *different* current value prices.

[2] Note that, even in the complete markets setting, debt may never be entirely repaid – or repayable – in finite time, although the date zero value of debt will tend to zero.

Finitely effective debt constraints are not the only debt constraints that will rule out Ponzi schemes. In a sense, however, they are the only debt constraints that are compatible with equilibrium and with the minimal ability to borrow and lend that we expect in our model. To make this assertion precise, we identify a broad class of debt constraints and show that whenever one of these more general debt constraints is compatible with an equilibrium, it necessarily reduces to the finitely effective constraints.

The class of debt constraints we consider is suggested by the following observations. The most straightforward way to repay current debt is to convert all future endowments into current wealth; when markets are complete, it is of course possible to accomplish this directly. When markets are incomplete, however, future endowments cannot be exchanged directly for current wealth; the 'optimal' strategy for converting future endowments into wealth today may involve borrowing at many future dates/events. Thus there is no unambiguous way to require that current debt can be repaid without simultaneously specifying what constraints debt must satisfy at these (subsequent) dates events.

This suggests that we should view debt constraints as an entire *system*, and specify debt constraints *simultantaneously* at all dates events, rather than *individually* at each date/event. [3] Given such a system of debt constraints, an equilibrium consists of a list of asset prices, commodity prices, consumption plans, and portfolio plans, such that the plans satisfy the usual market-clearing conditions and budget constraints *and* the given debt constraints, and are utility optimal among all such plans.

We are interested in systems of debt constraints that satisfy two conditions. Roughly speaking, a system of debt constraints is *loose* if liabilities that satisfy tomorrow's debt constraints can be acquired today. A system of debt constraints is *consistent* if liabilities that do not exceed today's debt constraint can be satisfied without exceeding tomorrow's debt constraints. To say that a system of debt constraints is both loose and consistent is to say that the debt constraint at each date event reflects an accurate summary of relevant information about future debt constraints. In the finite horizon setting, the implicit debt constraints are loose and consistent, and are the only such debt constraints. Thus, the notion of an equilibrium relative to any system of loose consistent debt constraints reduces to the usual one in the finite horizon setting.

The finitely effective debt constraints are always loose and consistent. In general, debt constraints which are loose and consistent need not be finitely effective. However, if a system of debt constraints is loose and consistent *and is*

[3] We formulate debt constraints as the spot value of debt that a trader may hold entering a particular date/event; this has the effect of constraining the portfolios that a trader may acquire at the preceding node.

compatible with some equilibrium, [4] then the debt constraints are *necessarily* finitely effective. In particular, these two notions of equilibrium coincide.

Another way to rule out Ponzi schemes is to limit not the level of debt at *each* node, but rather the *asymptotic behavior* of debt; the most obvious requirement to impose is that debt remains uniformly bounded (over all nodes). Remarkably, such a requirement again leads exactly to a finitely effective equilibrium.

In addition to these notions of equilibrium, there are other notions that do not fit easily into the framework of debt constraints. One way to capture these more general notions of equilibrium is by placing requirements directly on the intertemporal budget sets. Once again, these requirements lead exactly to a finitely effective equilibrium.

In addition to debt constraints, there is an additional difficulty that we must face: because we treat real assets, the dividend matrix may fail to have constant rank. In this regard, the infinite horizon setting is no different from the finite horizon setting; see Hart (1975). In this paper we content ourselves to follow Duffie and Shafer (1985, 1986) and establish the existence of a pseudo-equilibrium. We conjecture that, as in the finite horizon setting, pseudo-equilibria will generically be equilibria, but the precise notion of genericity required seems to be a subtle one.

Because our main purpose here is to emphasize the role of debt constraints, we restrict ourselves to the case of short-lived assets. However, there would be only notational difficulties in allowing for assets that have long – but finite – lives. Allowing for infinitely-lived assets would likely complicate the proofs (but not the definition).

To prove the existence of a finitely effective equilibrium, and the equivalence of a finitely effective equilibrium with the other two notions of equilibrium, we find it convenient first to establish the existence of a pseudo-equilibrium relative to some system of loose, consistent debt constraints. The argument, which follows Levine (1989), is somewhat involved, but the basic idea is straightforward. Every suitable finite truncation of the economy has a pseudo-equilibrium (with no debt constraints other than those implied by the constraint that there be no liabilities following the terminal date). The limit of these finite horizon pseudo-equilibria provides a pseudo-equilibrium for the infinite horizon economy, in which the debt constraints are taken to be the limit of the implicit debt constraints for the finite horizon truncations. We then show that equilibrium debt constraints that are loose and consistent are necessarily bounded below, and that consistent debt constraints that are bounded below can necessarily be repaid in finite time. Combining these results yields the existence of a finitely effective equilibrium and the equivalence with equilibrium relative to a system of loose, consistent debt constraints. Finally,

[4] Or pseudo-equilibrium; see below.

a variation on the same arguments establishes the equivalence with bounded debt equilibrium.

For short-lived numeraire assets (that is, assets denominated in a single commodity), pseudo-equilibria are necessarily equilibria, so in this case we obtain the existence of a true equilibrium. [5] Since the case of short-lived financial assets (that is, assets denominated in units of account) can be reduced to the case of numeraire assets, we obtain a true equilibrium in this case as well. [6]

Our approach to debt constraints is certainly not the only one possible, and three recent papers dealing with infinite horizon economies with incomplete asset markets treat debt constraints in quite a different way. Hernandez and Santos (1991) and Magill and Quinzii (1992) restrict the current value of debt to be no greater than the current value of future endowments. Hernandez and Santos compute current values with respect to a special system of current value prices, while Magill and Quinzii use personalized current value prices. Santos and Woodford (1993) insist that the current value of debt be no greater than the current value of future endowments, computed with respect to *all* current value prices. In Section 5 we show that these notions of equilibrium all coincide with a finitely effective equilibrium.

Our attention here is on infinite horizon economies populated by (a finite number of) infinitely lived traders. In an infinite horizon economy populated by finitely lived traders – for example, an overlapping generations economy – the issue of debt constraints can be resolved exactly as in the finite horizon setting: each individual faces the constraint that he cannot have liabilities after the terminal period of his life; debt at other dates is constrained implicitly by this requirement and by the budget constraints. For the existence of an equilibrium in an overlapping generations economy (with purely financial assets), see Schmachtenburg (1989).

2. Infinite horizon economies

Time and uncertainty are represented by a (countably) infinite tree S. A node $s \in S$ represents a finite history of exogenous events; we denote by $t(s)$ the length of that history. The root of the tree is denoted by $s = 0$; thus $t(0) = 0$. The node immediately preceding s is denoted by $s - 1$, and the set of nodes immediately following s is denoted by s^+.

[5] The restriction to short-lived assets is important here; the rank difficulty identified by Hart can occur even for numeraire assets that are long-lived.

[6] We do not consider economies with long-lived financial assets, but the existence of an equilibrium for such economies should not be problematical.

There are L commodities $1, \ldots, L$ available at each node. We write $p_s \in \Re^L_+$ for the vector of commodity spot prices at the node s, p_{sl} for the price of commodity l at s, and $p: S \to \Re^L_+$ for the function that assigns commodity spot prices at each node. It is convenient to normalize so that the value of the social endowment is 1 at each node. [7] A *consumption plan* is a bounded function $x: S \to \Re^L_+$; so $X = (l^\infty_+)^L$ is the consumption set (for each trader). [8] We write x_s for the vector of consumption at node s, and x_{sl} for the consumption of commodity l.

There are I traders $1, \ldots, I$ characterized by endowments $w^i \in X$ and utility functions $U^i: X \to \Re$. We assume that endowments and utility functions satisfy the following assumptions.

Assumption 1. The utility functions U^i are concave, monotonically increasing, and continuous in the Mackey topology. [9]

Assumption 2. Endowments are strictly positive and commensurable, in the sense that there is a constant $\rho > 0$ such that $w^j_s \geq \rho w^i_s$ for each node s and each pair of traders i, j.

Monotonicity and concavity are standard assumptions. Continuity in the Mackey topology is an assumption about time preference: additional consumption today is more desirable than additional consumption in the distant future. Time-discounted preferences certainly satisfy this assumption; for a detailed discussion, see Brown and Lewis (1981). The assumptions that endowments are strictly positive and commensurable serve three functions: they guarantee that some short selling is always possible (independent of prices); that income is strictly positive; and that debt constraints for different traders are commensurable (in the same sense that endowments are commensurable). [10,11]

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[7] We emphasize that the prices p_s are *spot* prices, *not* present value prices.

[8] The restriction to bounded consumption plans is innocuous; after a re-scaling, we may always assume that the social endowment is bounded, whence all feasible consumption plans are bounded. Of course, traders do not take social feasibility into account when they choose optimal plans. However, under extremely mild conditions, if a trader finds that a given bounded consumption plan is dominated by an unbounded consumption plan (satisfying appropriate constraints), then it will also be dominated by a bounded consumption plan (satisfying the same constraints). See Bewley (1972) for a similar discussion.

[9] Recall that, on bounded sets, the Mackey topology coincides with the product topology, and that – given our assumptions – the set of feasible consumption plans is bounded.

[10] It would actually suffice for our purposes to know that initial wealths are commensurable. Such a condition is of course implied by commensurability of endowments, and by various other assumptions.

[11] Note that endowments are commensurable if they are interior to l^∞_+, so our assumption is weaker than that of Bewley (1972).

Intertemporal transactions and insurance are carried out through the trade of short-lived (one-period) assets. For notational convenience, we assume that the number of assets M available at node s is independent of s. We write $q_s \in \Re^M$ for the vector of *asset prices* at node s, q_{sm} for the price of asset m at s, and $q : S \to \Re^M$ for the function that assigns asset prices to nodes. The portfolio of assets chosen by trader i at node s is denoted by y^i_s. A *portfolio plan* $y : S \to \Re^M$ assigns a portfolio choice at each node s.

We treat *real* assets, so that each asset purchased at node s returns a vector of commodities at each node $\sigma \in s^+$. We write R_σ for the *returns operator* at node σ; thus, if y_s is the portfolio held at the end of node s, then $R_\sigma y_s$ is the commodity bundle promised by portfolio y_s at node $\sigma \in s^+$.

We make two assumptions about asset returns.

Assumption 3 (Positive returns). For each node s there is a portfolio $y_s \geq 0$ such that $R_\sigma y_s \geq 0$ and $R_\sigma y_s \neq 0$ for each node $\sigma \in s^+$.

Assumption 4 (No redundant assets). For each node s and each pair of portfolios $y_s \neq y'_s$, there is a node $\sigma \in s^+$ such that $R_\sigma y_s \neq R_\sigma y'_s$.

The first of these assumptions provides a financial connection between dates. The second is purely for notational convenience; of course, redundant assets can be priced by arbitrage.

To motivate our final assumption, it will be useful to establish a lemma about intertemporal substitution. We first introduce some notation. Let x and y be consumption plans, c be a real number, and s be a node. By the *splice* $\langle x, c, y \mid s \rangle$ we mean the consumption plan defined by

$$\langle x, c, y \mid s \rangle_\tau = \begin{cases} c\mathbf{1} = (1, \ldots, 1), & \text{if } \tau = s, \\ y_\tau, & \text{if } \tau \text{ follows } s, \\ x_\tau, & \text{otherwise.} \end{cases}$$

Lemma A. For each trader i, feasible consumption plan x^i, and node s, there are real numbers c, r, and δ with $c > 0$, $0 < r < 1$, and $0 < \delta < 1$, such that the consumption plan $(1-\delta)x^i + \delta\langle x^i, c, rw^i \mid s \rangle$ is preferred to x^i.

Proof. Concavity implies that U^i has right-hand directional derivatives at x^i in every direction. We claim that, for c sufficiently large, the right-hand derivative (call it β_c) of U^i at x^i in the direction $\langle x^i, c, w^i \mid s \rangle - x^i$ is strictly positive. Assuming this claim, the remainder of the argument is simple. For $\delta > 0$, the definition of the right-hand derivative yields:

$$\begin{aligned} U^i((1-\delta)x^i + \delta\langle x^i, c, w^i \mid s \rangle) &= U^i(x^i + \delta[\langle x^i, c, w^i \mid s \rangle - x^i]) \\ &= U^i(x^i) + \beta_c \delta + o(\delta), \end{aligned}$$

where $o(\delta)/\delta \to 0$ as $\delta \to 0$. Since $\beta_c > 0$, we conclude that

$$U^i((1-\delta)x^i + \delta\langle x^i, c, w^i \mid s\rangle) > U^i(x^i),$$

provided that δ is sufficiently small. Continuity then implies that

$$U^i((1-\delta)x^i + \delta\langle x^i, c, rw^i \mid s\rangle) > U^i(x^i),$$

provided that r is sufficiently close to 1, as asserted.

It remains to establish the claim. We write $x^* = \langle x^i, 0, w^i \mid s\rangle - x^i$ and $z = \langle 0, 1, 0 \mid s\rangle$, so that $\langle x^i, c, w^i \mid s\rangle - x^i = x^* + cz$. Concavity implies that the right-hand derivatives are at least as large as the difference quotients:

$$\beta_c \geq \frac{U^i(x^i + \epsilon[x^* + cz]) - U^i(x^i)}{\epsilon}$$

for each $\epsilon > 0$. Setting $\epsilon = 1/c$ yields:

$$\beta_c \geq \frac{U^i(x^i + (1/c)[x^* + cz]) - U^i(x^i)}{1/c}$$
$$= c\left[U^i(x^i + (1/c)x^* + z) - U^i(x^i)\right].$$

Since U^i is monotonically increasing and is continuous at x^i, it follows that $U^i(x^i + (1/c)x^* + z) - U^i(x^i) > 0$ if c is sufficiently large, so that

$$\beta_c \geq c\left[U^i(x^i + (1/c)x^* + z) - U^i(x^i)\right] \to \infty$$

as $c \to \infty$. In particular, $\beta_c > 0$ for c sufficiently large. □

Lemma A says that, for each node s, there is a level of consumption sufficiently large that a convex combination of this additional consumption and a little consumption in the future is an improvement. Our final assumption simply posits that this level of consumption can be chosen uniformly, independently of the particular node.

Assumption 5. For each trader i, there are real numbers c and r, with $c > 0$ and $0 < r < 1$, with the property:

> for each feasible consumption plan x^i and node s, there is a real number δ, with $0 < \delta < 1$, such that the consumption plan $(1-\delta)x^i + \delta\langle x^i, c, rw^i \mid s\rangle$ is preferred to x^i.

It is important to keep in mind that this is an assumption about utility functions *and* endowments (and hence about feasible consumptions). Since we have assumed endowments are bounded, this assumption is satisfied for stationary discounted expected utility functions – that is, functions of the form

$$U^i(x^i) = \sum_t \delta^t \sum_{t(s)=t} \pi_t(s) u^i(x^i(s)),$$

where $\delta < 1$ is a discount factor and π_t is a probability distribution on the set of nodes occurring at time t. However, it fails for non-stationary utility functions of the form

$$U^i(x^i) = \sum_t \delta^t \sum_{t(s)=t} \pi_t(s)\left[1 - \exp\left(-(1/t)\sum x_l^i(s)\right)\right]$$

whenever endowments are bounded away from zero. [12]

Given commodity spot prices p_σ, the portfolio y_s yields a *dividend* of $p_\sigma \cdot R_\sigma y_s$ (units of account) at the node σ. It is convenient to write $V_s(p)$ for the *dividend operator* which maps portfolios at the node s to the vector of dividends at nodes in s^+:

$$(V_s(p)y_s)(\sigma) = p_\sigma \cdot R_\sigma y_s.$$

Since there are M assets, the dividend operator has rank at most M, but it may have lower rank for some prices.

No production or intertemporal storage is possible, and assets are in zero net supply, so the social feasibility conditions for the economy are

$$\sum_i x_s^i \le \sum_i w_s^i \quad \text{and} \quad \sum_i y_s^i = 0.$$

Initial holdings of securities are zero. It is convenient to write $y_{-1}^i = 0$. Thus, for every node s, trader i faces a budget constraint which may be written as

$$p_s \cdot \left(x_s^i - w_s^i\right) + q_s \cdot y_s^i \le p_s \cdot R_s y_{s-1}^i.$$

(Note that this inequality is homogeneous in (p_s, q_s), so that we are indeed free to normalize so that the social endowment has value 1 at each node.) The *pre-budget set* is the set $B^i(w^i, p, q)$ of consumption and portfolio plans (x^i, y^i) that satisfy this budget constraint at each node.

As we have noted, the constraints imposed to this point are not sufficient to rule out unbounded levels of borrowing. To rule out such Ponzi schemes we need restrictions on the budget sets. We consider three such restrictions. The first requires that, at each node, it should be possible to repay *almost all* the debt in finite time; this is an endogenous requirement. The second limits debt at each node by an exogenously given system of debt constraints. The third limits only the asymptotic nature of debt; this is again an endogenous requirement.

It is convenient to derive the first two restrictions from a common framework. A system of *debt constraints* for trader i is a function

$$D^i : S - \{0\} \to [-\infty, 0].$$

[12] This example is of some significance: if we begin with an economy in which endowments grow linearly with time and rescale so that endowments become bounded, then a stationary exponential utility function is transformed into one of this form.

Given commodity prices p, the portfolio $y_s \in \Re^M$ *satisfies the debt constraint* at $\sigma \in s^+$ if

$$V_\sigma(p) y_s = p_\sigma \cdot R_\sigma y_s \geq D^i_\sigma .$$

The interpretation we have in mind is simple. If trader i holds portfolio y_s at the end of node s, then he will owe a debt (liability) of $V_\sigma(p) y_s$ at each node $\sigma \in s^+$; a debt constraint limits this debt, and hence implicitly limits the set of portfolios that can be held at s. We write $Y_s \subset \Re^M$ for the set of portfolios y_s that satisfy the debt constraint at each node $\sigma \in s^+$. [13]

Of course, the role of debt constraints is to rule out (some) Ponzi schemes in the infinite horizon economy. Notice that debt limits are non-positive – i.e. traders cannot be forced to save. As we have formulated them, debt constraints could be identically $-\infty$; of course, such debt constraints are not compatible with any equilibrium. As we shall see, compatibility with an equilibrium (together with other requirements we impose below) forces debt constraints to be finite everywhere.

It is instructive to think about the role of debt constraints in the finite horizon framework. In that framework, debt cannot be held at the end of the terminal period, and this constraint gives rise to implicit debt constraints at earlier nodes as well. The budget constraint forces repayment by the terminal date, so the debt limit at any node s is the greatest amount of debt that the trader could hold, entering node s, and still be able to repay by the terminal date. In the infinite horizon framework there is no terminal constraint, so it is necessary to impose a system of debt constraints to make these implicit constraints explicit.

We find it convenient to express debt constraints in terms of the value of the portfolio held at the beginning of the period, rather than at the end of the period. To understand why, consider the implicit debt constraints in the finite horizon model. With incomplete markets, the amount of debt that can be held at the end of the period depends on the form in which it is held. If a trader is short in securities that promise repayment in future states in which his endowment is large, then a substantial debt can be repaid; if he is short in securities that promise repayment in future states in which his endowment is small, then he can repay very little. If debt were defined in terms of end-of-period holdings, then it would be necessary to distinguish various portfolios of debt. Our definition in terms of beginning-of-period holdings is therefore convenient because it enables us to work entirely in terms of value.

Given endowments w^i, prices p and q, and debt constraints D^i, the consumption/portfolio plan (x^i, y^i) belongs to the *budget set* $B^i(w^i, p, q, D^i)$ for trader i if, for each node, s:

[13] Note that there is no debt constraint at the initial node since there are no portfolio choices prior to the initial node.

- the budget constraint is satisfied at s, i.e.

$$p_s \cdot (x_s^i - w_s^i) + q_s \cdot y_s^i \le p_s \cdot R_s y_{s-1}^i;$$

- the debt constraint is satisfied at s, i.e.

$$V_s(p) y_{s-1} = p_s \cdot R_s y_{s-1} \ge D_s^i.$$

In this circumstance we frequently say that the portfolio plan y^i *finances* the consumption plan x^i.

An *equilibrium relative to the debt constraints* (D^i) consists of prices p and q, consumption plans (x^i) and portfolio plans (y^i) such that

- consumption plans are socially feasible;
- portfolio plans are socially feasible (i.e. $\Sigma_i y_s^i = 0$ for each s); and
- for each trader i, x^i maximizes trader i's utility over all plans belonging to the budget set $B^i(w^i, p, q, D^i)$

Unfortunately, an equilibrium may not exist in this setting. As noted in the Introduction, the difficulty is a technical one, familiar from the finite horizon setting: for some prices p, the dividend operator $V_s(p)$ may have rank less than M. We shall be content in general to obtain a pseduo-equilibrium (in two cases we shall obtain a true equilibrium); in this we follow Duffie and Shafer (1985, 1986). Our formulation is different from theirs (we use spot prices rather than present value prices), although the formulations are in fact equivalent.

For each node s, we consider an M-dimensional subspace $K_s \subset \Re^{s^+}$ of *income transfers*, and a *pricing functional* $Q_s : K_s \to \Re$. An *income transfer plan* is a family of vectors $k_s \in K_s$. For $\sigma \in s^+$, we write $k_s(\sigma)$ for the σ-component of k_s. Given commodity prices p, the consumption/income transfer plan (x^i, k^i) *satisfies the budget constraint* at s if

$$p_s \cdot (x_s^i - w_s^i) + Q_s \cdot k_s \le k_{s-1}(s).$$

As before, the *pre-budget set* $B^i(w^i, p, K, Q)$ is the collection of consumption/income transfer plans (x^i, k^i) that satisfy the budget constraint at each node. Similarly, (x^i, k^i) *satisfies the debt constraint* at s if

$$k_{s-1}(s) \ge D_s^i.$$

Finally, (x^i, k^i) belongs to the *budget set* $B^i(w^i, p, K, Q, D^i)$ for trader i if it satisfies the budget and debt constraints at each node. Again, we frequently say that the income transfer plan k^i *finances* the consumption plan x^i.

A *pseudo-equilibrium relative to the debt constraints* (D^i) consists of prices p, a family K of subspaces of income transfers, pricing functionals Q, consumption plans (x^i), and income transfer plans (k^i) such that

- consumption plans are socially feasible;
- income transfer plans are socially feasible (i.e. $\Sigma_i k_s^i = 0$ for each s);
- for each trader i, the plan (x^i, k^i) maximizes trader i's utility over all plans belonging to the budget set $B^i(w^i, p, K, Q, D^i)$; and
- for each s, the range of the dividend operator $V_s(p)$ is a subspace of K_s.

A pseudo-equilibrium is *proper* if, for each s, the range of the dividend operator $V_s(p)$ is equal to K_s.

If trader i acquires portfolio y_s at node s, he will affect the income transfers $V_\sigma(p)y_s$ at nodes $\sigma \in s^+$. Since the definition of a pseudo-equilibrium requires that the range of the dividend operator $V_s(p)$ be a subspace of the space K_s of income transfers, allowing income transfers to lie in K_s expands the possibilities for each trader. Thus, the notion of a pseudo-equilibrium is more general than the notion of an equilibrium. Moreover, proper pseudo-equilibria are actually equilibria. More precisely, if

$$\langle p, K, Q, (x^i), (k^i) \rangle$$

is a proper pseudo-equilibrium relative to the debt constraints (D^i), then there are asset prices q and portfolio plans (y^i) such that

$$\langle p, q, (x^i), (y^i) \rangle$$

is an equilibrium relative to the the debt constraints (D^i). To see this, we need only note that the pricing functional Q_s defines prices q_s for asset portfolios by the rule

$$q_s \cdot y_s = Q_s \cdot V_s(p),$$

and that the income transfer plans k^i define portfolio plans y^i by the rule

$$V_s(p)y_s^i = k_s^i.$$

It is straightforward to verify the equilibrium conditions.

The debt constraints of most interest to us are those we call finitely effective. To define them, we proceed in the following way. Fix a system of commodity spot prices p and pricing functionals Q. Consider a trader i, a particular node s, an amount of debt $d < 0$, and a finite time horizon T. We say that debt d can be *repaid in T periods from node s* if there is a consumption plan x^i and an income transfer plan k^i which meet the given debt d at node s, are budget feasible at every node σ which follows s by fewer than T periods, and leave 0 debt after T periods. Formally:

- $p_s \cdot x_s^i + Q_s \cdot k_s^i - d \le p_s \cdot w_s^i$;
- $p_\sigma \cdot x_\sigma^i + Q_\sigma \cdot k_\sigma^i \le p_\sigma \cdot w_\sigma^i + k_{\sigma-1}^i(\sigma)$
  for every node σ that follows s and satisfies $t(s) < t(\sigma) < t(s) + T$; and
- $k_{\tau-1}^i(\tau) \ge 0$ for every node τ that follows s and satisfies $t(\tau) = t(s) + T$.

The debt $d < 0$ can be *repaid in finite time from node s* if it can be repaid in T periods for some T. The *finitely effective debt constraints* FE^i for trader i are defined by

$$FE_s^i = \inf\{d : d \text{ can be repaid in finite time from node s}\}$$

It is important to keep in mind that the finitely effective debt constraints are *uniquely and endogenously defined* (of course they depend on commodity prices and asset pricing functionals).

Note that our definition allows for the possibility that $FE^i_s = -\infty$ for some – indeed, for all – nodes s. As we have already noted, however, such debt constraints are incompatible with an equilibrium. Indeed, we shall show that finitely effective debt constraints that are compatible with an equilibrium (or a pseudo-equilibrium) are uniformly bounded below.

A *finitely effective pseudo-equilibrium* consists of commodity prices p, subspaces K, asset pricing functionals Q, consumption plans (x^i) and income transfer plans (k^i) that constitute a pseudo-equilibrium with respect to the debt constraints (FE^i). That is,

- consumption plans are socially feasible;
- income transfer plans are socially feasible;
- for each trader i, the plan (x^i, k^i) maximizes trader i's utility over all plans belonging to the budget set $B^i(w^i, p, K, Q, FE^i)$; and
- for each s, the range of the dividend operator $V_s(p)$ is a subspace of K_s.

Although finitely effective debt constraints seem quite natural, they are perhaps not the only debt constraints of interest. We restrict our attention here to debt constraints that satisfy two conditions:

- if debt can be acquired, then it can be repaid; that is, if the current debt constraint is satisfied, then there is a plan that meets today's liabilities and satisfies tomorrow's debt constraints;
- if debt can be repaid, then it can be acquired; that is, if there is a plan that meets a given liability today and satisfies tomorrow's debt constraints, then the given liability satisfies today's debt constraints.

To formalize the first requirement, we fix commodity spot prices p and pricing functionals Q. The debt constraint D^i is *consistent* at node s if for every income transfer plan $k^i_{s-1} \in K_{s-1}$ that meets the debt constraint at s – that is, $k^i_{s-1}(s) \geq D^i_s$ – there is an income transfer plan $k^i_s \in K_s$ such that

$$k^i_{s-1}(s) + p_s \cdot w^i_s - Q_s \cdot k^i_s \geq 0$$

and $k^i_s(\sigma) \geq D^i_\sigma$ for each $\sigma \in s^+$. Since the only requirement on the income transfer plan k^i_{s-1} is that it meet the debt constraint at node s, and it is always possible to find such a k^i_{s-1} such that $k^i_{s-1}(s) = D^i_s$, an alternative formulation of consistency is: there is a plan $k^i_s \in K_s$ such that

$$D^i_s + p_s \cdot w^i_s - Q_s \cdot k^i_s \geq 0$$

and $k^i_s(\sigma) \geq D^i_\sigma$ for each $\sigma \in s^+$. That is to say, it is possible to meet a liability equal to or greater than today's debt constraint by consuming nothing today and acquiring an income transfer (that is, borrowing) that meets tomorrow's debt constraints. The system D^i of debt constraints is *consistent* if it is consistent at each node.

Note that consistency of the entire system expresses the idea desired: if debt satisfies the limit at a particular node, and debt constraints are consistent at that node, then debt can be rolled over to satisfy the constraints next period. If the entire system of debt constraints is consistent, this process can be repeated, so constraints can be satisfied at every future node. In other words, the current constraint correctly summarizes future constraints.

The requirement that the system of debt constraints be consistent is important, but not limiting in itself: any given system of debt constraints – consistent or not – can be modified to a system that is consistent and yields exactly the same budget sets. (Recall that the definition of budget sets involves both budget constraints and debt constraints.)

However, merely to establish the existence of an equilibrium with some system of debt constraints – even some consistent system of debt constraints – does not seem very satisfactory. There is, for example, an equilibrium in which $D_s^i = 0$ for every trader i and node s, in which there is no intertemporal trade or insurance. (Zero debt constraints are clearly consistent.) In the finite horizon model, the usual (implicit) assumption is that if debt can be repaid, then it can be acquired; we want a similar property for the infinite horizon model as well.

To formalize our second requirement, we fix a node s and trader i. Consider the set K_s^i of income transfers $k_s \in K_s$ such that $k_s(\sigma) \geq D_\sigma^i$ for each node $\sigma \in s^+$; these are the income transfers that meet trader i's debt constraints at the next date. Selling the endowment w_s^i at the spot prices p_s and acquiring the income transfer $k_s^i \in K_s^i$ at the prices Q_s will generate revenue at s equal to $p_s \cdot w_s^i - Q_s \cdot k_s$, and hence will repay an amount of debt equal to $-p_s \cdot w_s^i + Q_s \cdot k_s$. To say that debt which can be repaid can be acquired is to say that all debts which can be repaid meet the debt limits at node s; in particular,

$$-p_s \cdot w_s^i + Q_s \cdot k_s \geq D_s^i.$$

We define the debt constraint D_s^i to be *loose* at node s if this inequality is satisfied for every income transfer $k_s \in K_s$ that satisfies the debt constraints $k_s(\sigma) \geq D_\sigma^i$ at each $\sigma \in s^+$. The system D^i is *loose* if it is loose at every node.

It is important to keep in mind that looseness and consistency are properties of a *system* of debt constraints, and that their validity depends on the particular system of commodity spot prices and asset pricing functionals.

It is easily seen that the finitely effective debt constraints are loose and consistent (because the plan which repays today's debt in T periods can be extended to repay yesterday's debt in $T+1$ periods). Indeed, the finitely effective debt constraints constitute the least upper bound of all possible systems of loose, consistent debt constraints. At the opposite extreme, the debt constraints that are identically $-\infty$ are loose and consistent; of course, debt constraints that are identically $-\infty$ cannot be consistent with any equilibrium. (Note that the debt constraints that are identically zero are consistent – but not loose.)

In the finite horizon setting, the implicit debt constraints (that is, debt at each node is constrained to the level that can be repaid by the terminal period) are finitely effective, and hence loose and consistent. In fact, the implicit debt constraints are the *only* debt constraints that are loose and consistent and repay all debt at the terminal nodes.

Because we are working entirely with spot prices, *we do not require* particular bounds on debt constraints, nor do we require asymptotic behavior in the distant future. As we shall show, however, our assumptions imply that loose and consistent debt constraints that are compatible with an equilibrium are necessarily bounded below. We return to this point in the following sections.

For a given system of prices, there may be many systems of debt constraints that are loose and consistent. Although this might seem troubling, it should not. After all, we are not interested in debt constraints per se, but rather in debt constraints *and equilibrium.* As we shall see, the requirement that a system of debt constraints be compatible with equilibrium implies that the debt constraints necessarily reduce to the finitely effective ones.

Either the finitely effective debt constraints or the more general loose and consistent debt constraints are sufficient to rule out Ponzi schemes. Another way to achieve this is to place limitations not on the debt at each node, but rather on the asymptotic behavior of debt. In particular, we consider the implications of requiring that debt be uniformly bounded across all nodes. We define the *bounded debt budget sets* $B^i_\beta(w^i, p, q,)$ to be the set of consumption and income transfer plans (x^i, k^i) that satisfy the budget constraint at each node

$$p_s \cdot \left(x^i_s - w^i_s\right) + Q_s \cdot k_s \leq k_{s-1}(s)$$

and have the additional property that debt is uniformly bounded below:

$$\inf\left\{k^i_s(\sigma) : s \in S,\ \sigma \in s^+\right\} > -\infty.$$

A *bounded debt pseudo-equilibrium* consists of commodity prices p, subspaces K, asset pricing functionals Q, consumption plans (x^i) and income transfer plans (k^i) such that:

- consumption plans are socially feasible;
- income transfer plans are socially feasible;
- for each trader i, the plan (x^i, k^i) maximizes trader i's utility over all plans belonging to the budget set $B^i_\beta(w^i, p, q,)$; and
- for each s, the range of the dividend operator $V_s(p)$ is a subspace of K_s.

It is perhaps useful at this point to briefly contrast our three notions of a pseudo-equilibrium. Each of these notions begins with the same pre-budget sets: the consumption/income transfer plans that satisfy the budget constraints at each node. A finitely effective pseudo-equilibrium is based on budget sets that incorporate the additional requirement that debt at each node be almost repayable in finite time. Note that this is an endogenous requirement, and that it represents an infinite family of constraints (one at each node). A pseudo-equilibrium with respect to a system of loose, consistent debt constraints is based on budget sets that incorporate the additional requirement that debt at each node be bounded by exogenously given constraints. Finally, a bounded debt pseudo-equilibrium is based on budget sets that incorporate the additional requirement that debt remains uniformly bounded; note that this is an endogenous requirement, but that it is an overall constraint (not a constraint at each node separately).

Our debt constraints are based on spot prices and on what a trader could repay, not on present value prices and wealth (that is, the present value of future endowment). A simple example, adapted from Hernandez and Santos (1991), may illustrate why this is an important distinction. Let us consider a tree S that has two branches at each node, so that each node s has two successors; we write this as $s^+ = \{s_0, s_1\}$. Let us assume that there is a single commodity available for consumption at each node, and a single one-period asset, which promises delivery of one unit of consumption at each successor node. We consider a trader i whose endowment w^i is given by $w^i(0) = 0$ and $w^i(s_0) = 0$, $w^i(s_1) = 1$ for each node s^+. If (present value) prices are strictly positive, then this trader's wealth (that is, the present value of his future endowment) is strictly positive at each node. However, the finitely effective debt constraints are identically zero at each node; with such debt constraints, no borrowing is possible at any node. Such zero debt constraints are perfectly sensible here: in any finite horizon truncation of this tree it will be impossible for this trader to borrow at any node, since it might be impossible for him to repay his debt by the terminal node. The implicit debt constraints in these finite horizon truncations are therefore identically zero at each node. Zero debt constraints in the infinite horizon setting therefore correctly capture the finite horizon limit.

Of course, we do not assert that a sensible theory of debt constraints cannot be based on present values. Our theory is not, and the distinction is a real one. In fact, Hernandez and Santos (1991) and Magill and Quinzii (1992) offer notions of an equilibrium that *are* based on (individualized) present values – although we show in Section 5 that those notions of equilibrium do not differ from ours.

3. A pseudo-equilibrium

Our fundamental results are:

Theorem 1. Every infinite horizon economy satisfying Assumptions 1–5 admits a finitely effective pseudo-equilibrium.

Theorem 2. In the presence of Assumptions 1–5, the notions of a finitely effective pseudo-equilibrium, of a pseudo-equilibrium relative to some system of loose, consistent debt constraints, and of a bounded debt pseudo-equilibrium, are equivalent.

In the finite horizon setting, Duffie and Shafer (1985, 1986) have shown (with the additional assumption of smooth preferences) that, generically in endowments and asset structure, pseudo-equilibria are in fact equilibria. We conjecture that a similar result holds in our setting; however, giving a precise meaning to 'generically in endowments and asset structure' does not seem an easy task in the infinite horizon context. We are content here to show that we can obtain a true equilibrium in two cases: if all assets are denominated in a single commodity (*numeraire assets*), or if all assets are denominated in units of account (*financial assets*).

Corollary 1. If all assets are denominated in a single commodity, then there is a finitely effective equilibrium.

Corollary 2. If all assets are denominated in units of account, then there is a finitely effective equilibrium.

The proofs of the theorems are rather involved, and so we defer them to the following section. However, the corollaries have quite simple proofs. To obtain the first of these corollaries, observe that, for one-period numeraire assets, the returns operator necessarily has constant rank M at each node, so that the notions of an equilibrium and a pseudo-equilibrium coincide. To obtain the second of these corollaries, observe that purely financial assets can be reinterpreted as securities denominated in units of the social endowment (which, according to our normalization, always has price 1), and that the returns operator again has constant rank M at each node (so that the notions of an equilibrium and a pseudo-equilibrium coincide).

4. Proofs

We find it convenient to isolate several lemmas. The first establishes the existence of a pseudo-equilibrium relative to some system of loose, consistent debt constraints. Note that this lemma, and the next, use Lemma A, but not Assumption 5 (which expresses the uniform version of the conclusion of Lemma A).

Lemma B. Every infinite horizon economy satisfying Assumptions 1–4 admits a pseudo-equilibrium relative to some system of loose, consistent debt constraints.

Proof. We construct a pseudo-equilibrium for our infinite horizon economy as a limit of pseudo-equilibria for appropriate finite horizon truncations. To this end, we fix a time horizon T and consider the finite horizon economy $\mathscr{E}(\mathscr{T})$ obtained in the following way:

- time and uncertainty are described by the tree $S(T)$ that consist of all nodes $s \in S$ for which $t(s) \leq T$;
- the commodities and assets available for trade at each node of $S(T)$ are the same as at the corresponding node of S, except that no assets are available at terminal nodes of $S(T)$;

- there are I traders; endowments at each node of $S(T)$ are the same as at the corresponding node of S;
- trader i's utility $\overline{U}_T^i(x^i)$ for the consumption plan $x: S(T) \to \Re_+^L$ is set equal to his utility for the plan x^* which coincides with x at each node $s \in S(T)$ and with w_s^i at each node $s \notin S(T)$.

According to Geanakoplos and Shafer (1990), the finite horizon economy $\mathscr{E}(\mathscr{T})$ has a pseudo-equilibrium

$$E(T) = \langle p(T), K(T), Q(T), (x^i(T)), (k^i(T)) \rangle$$

with no debt constraints (other than the terminal ones). [14]

We now let $T \to \infty$ and pass to a convergent subsequence. To do this, we must first verify that all the various components of the pseudo-equilibrium $E(T)$ lie in compact sets. For some of these components, this is trivial:

- commodity prices $p_s(T)$ are bounded (since the value of the social endowment is 1);
- subspaces $K_s(T)$ of income transfers lie in the compact Grassman manifold of M-dimensional subspaces of $\Re^{s^+}$; and
- consumption vectors $x_s^i(T)$ are non-negative and bounded by aggregate endowments

Passing to a subsequence, if necessary, we write p_s for the limit commodity spot prices, K_s for the limit subspaces of income transfers, and x_s^i for the limit consumption vectors.

(a) Income transfers $k_s^i(T)$ are bounded above. For, if not, we could find a node $\sigma \in s^+$ for which $k_s^i(T)(\sigma)$ is unbounded above. For real numbers c and δ, consider the consumption plans z^i and $z^i(T)$ defined by

$$z^i = (1-\delta)x^i + \delta\langle x^i, c, w^i \mid s\rangle,$$

$$z^i(T) = (1-\delta)x^i(T) + \delta\langle x^i(T), c, w^i \mid s\rangle.$$

According to Lemma A (in Section 2), we can choose c and δ so that z^i is preferred to x^i. The continuity of utility functions in the product topology entails that

$$U^i(z^i(T)) \to U^i(z^i),\ U^i(x^i(T)) \to U^i(x^i).$$

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[14] Geanakoplos and Shafer formulate a pseudo-equilibrium in terms of present value prices, rather than spot prices, but the notions are equivalent for finite horizon economies. They also assume that the indifference surface through any interior consumption plan is a closed subset of the strictly positive orthant, an assumption that we have not made. However, this assumption is unnecessary. To see this, let $\overline{U}$ be any quasi-concave utility function having the desired indifference surfaces; for each $\epsilon > 0$, consider the utility functions $U_\epsilon^i = U^i + \epsilon\overline{U}$. Evidently, the utility functions U_ϵ^i also have the desired indifference surfaces. We write $\mathscr{E}_\epsilon(T)$ for the economy obtained by substituting these utility functions. Applying the result of Geanakoplos and Shafer, we conclude that $\mathscr{E}_\epsilon(T)$ has a pseudo-equilibrium. Letting $\epsilon \to 0$, and passing to the limit (of a subsequence, if necessary) we obtain a pseudo-equilibrium for the economy $\mathscr{E}(T)$.

Hence, $z^i(T)$ is preferred to $x^i(T)$ for T sufficiently large. Since $k_s^i(T)(\sigma)$ is unbounded above, the consumption plan $\langle x^i(T), c, w^i \mid s\rangle$ is budget feasible if T is sufficiently large. [15] Hence $z^i(T)$ is a convex combination of budget feasible plans, and therefore is itself budget feasible for T sufficiently large. This is a contradiction, so we conclude that income transfers are indeed bounded above.

(b) Income transfers $k_s^i(T)$ are bounded below, since they are bounded above, and the sum of income transfers of all traders is identically 0.

(c) Prices $Q_s(T)$ are non-negative and bounded above. Non-negativity is clear, since preferences are increasing. If prices $Q_s(T)$ are not bounded above, we may choose, for each T, a trader $i(T)$ such that $k_{s-1}^{i(T)}(T)(s) \geq 0$; for notational convenience, in what follows we suppress the dependence of i on T. Again, we can use Lemma A to choose real numbers c, r, and δ, with $c > 0$, $0 < r < 1$, and $0 < \delta < 1$, and define a consumption plan

$$z^i = (1-\delta)x^i + \delta\langle x^i, c, rw^i \mid s\rangle$$

which is preferred to x^i. We set

$$z^i(T) = (1-\delta)x^i(T) + \delta\langle x^i(T), c, rw^i \mid s\rangle.$$

Continuity guarantees that $z^i(T)$ is preferred to $x^i(T)$, provided that T is sufficiently large. However, if T is sufficiently large, then the consumption plan $\langle x^i(T), c, rw^i \mid s\rangle$ is budget feasible. To see this, we note first that since only a finite number of assets A are available at s, their payoffs at nodes in s^+ are bounded by some multiple of the consumption vector $\mathbf{1} = (1, \dots, 1)$; say $A(\sigma) \leq \alpha\mathbf{1}$, for each asset A. Endowments are strictly positive, so $w^i > \beta\mathbf{1}$ for some $\beta > 0$. We choose a real number ϵ with $0 < \epsilon < (1-r)\beta/\alpha$. We have assumed that prices $Q_s(T)$ are unbounded above, so, for T sufficiently large, there is an asset A^* whose price is at least cL/ϵ. The consumption plan $\langle x^i(T), c, rw^i \mid s\rangle$ can then be financed by the following plan of income transfers: at node s, sell ϵ units of asset A^* (this yields income sufficient to purchase $c\mathbf{1}$); do nothing at nodes following s (liabilities at $\sigma \in s^+$ arising from the sale of A^* at s can be covered by the fraction of endowment $(1-r)w_\sigma^i$), and follow the income transfer plan $k_\tau^i(T)$ at every other node τ. Thus, the consumption plan $z^i(T)$ is the convex combination of budget feasible plans, and therefore is itself budget feasible if T is sufficiently large. This is a contradiction, so we conclude that prices $Q_s(T)$ are bounded above, as asserted.

Now that we have established that the components of the equilibria $E(T)$ lie in compact sets, we may extract a subsequence converging to

$$E = \langle p, K, Q, (x^i), (k^i)\rangle.$$

[15] That is, this is the consumption part of a consumption/income transfer plan in trader i's budget set.

The next step is to construct suitable debt constraints as limits of implicit debt constraints for each of the economies $\mathscr{E}(T)$. To this end, we fix a trader i, a node s and an index $T > t(s)$. We define the implicit debt constraint $D_s^i(T)$ for trader i in economy $\mathscr{E}(T)$ as

$$D_s^i(T) = \inf\{-p_s \cdot (\bar{x}_s^i - w_s^i) - Q_s \cdot \bar{k}_s^i\},$$

where the infimum is taken over all consumption and income transfer plans $(\bar{x}^i, \bar{k}^i)$ that meet the budget constraints (relative to commodity prices $p(T)$ and pricing functionals $Q(T)$) at s and at every node τ following s. [16]

(d) The implicit debt constraints $D_s^i(T)$ are bounded below (at each node). If not, suppose that trader i's implicit debt constraints are not bounded below at node s. For each T, choose a trader j for whom $k_{s-1}^j(T)(s) \geq 0$. (We suppress the dependence of j on T.) According to Assumption 2, there is a real number $\rho > 0$ such that $w_s^j \geq \rho w_s^i$ for each trader j. If we apply Lemma A again, then we may find real numbers c, r, and δ, with $c > 0$, $0 < r < 1$, and $0 < \delta < 1$, so that the consumption plan

$$z^j = (1 - \delta) x^j + \delta \langle x^j, c, rw^j \mid s \rangle$$

is preferred to x^j. Continuity implies that

$$z^j(T) = (1 - \delta) x^j(T) + \delta \langle x^j(T), c, rw^j \mid s \rangle$$

is preferred to $x^j(T)$ if T is sufficiently large. We assert that $z^j(T)$ is budget feasible if T is sufficiently large. To establish this, it is sufficient to show that $\langle x^j(T), c, rw^j \mid s \rangle$ is budget feasible if T is sufficiently large (since $z^j(T)$ is a convex combination of $\langle x^j(T), c, rw^j \mid s \rangle$ and the equilibrium consumption $x^j(T)$). By definition of the implicit debt constraint $D_s^i(T)$, there is an income transfer plan $h^i(T)$ for trader i that, beginning at the node s, repays the debt $D_s^i(T)$ (provided that trader i consumes nothing at subsequent nodes). In other words, the plan $h^i(T)$ yields income $-D_s^i(T)$ at node s, and involves no liabilities at the nodes at time T. By Assumption 2, the endowments of trader i and trader j are commensurable: $w^j \geq \rho w^i$. Hence, by following the plan $\rho h^i(T)$, beginning at node s, trader j can obtain income $-\rho D_s^i(T)$ at node s, and still meet all his liabilities at subsequent nodes (provided he consumes nothing). And if trader j follows the plan $h^j = (1 - r)\rho h^i(T)$, beginning at node s, he can obtain an income of $-(1 - r)\rho D_s^i(T)$ at node s, consume the portion rw^j of his endowment at all subsequent nodes, and still meet all his liabilities. We define the income transfer plan H^j by $H_\tau^j = h_\tau^j$ if $\tau = s$ or τ follows s, and $H_\tau^j = k_\tau^j(T)$ for all other nodes τ. This income transfer plan finances the consumption plan $\langle x^j(T), c, rw^j \mid s \rangle$, provided that the income it generates at node s is sufficient to purchase the consumption bundle $c\mathbf{1}$. The income generated by H^j at s is equal to

[16] We make no restrictions on $(\bar{x}^i, \bar{k}^i)$ at other nodes.

$-(1-r)\rho D_s^i(T)$. Since we have assumed that $D_s^i(T)$ is unbounded below, we conclude that the consumption plan $\langle x^j(T), c, rw^j \mid s\rangle$ is budget feasible, provided that T is sufficiently large. But then $z^j(T)$ is budget feasible and preferred to $x^j(T)$, which is a contradiction. We conclude that implicit debt constraints are bounded below.

Now that we have established that the implicit debt constraints are bounded below, we may, passing to a subsequence if necessary, assume that

$$D_s^i(T) \to D_s^i$$

for each trader i and node s. This provides us with with a system of debt constraints; it remains to show that these debt constraints are loose and consistent, and that the tuple E constitutes a pseduo-equilibrium relative to these debt constraints.

It is trivial to verify that that individual consumption plans and transfer plans belong to the individual budget sets at each node, that consumption plans and income transfer plans are socially feasible, and that the range of each dividend operator lies in the appropriate income transfer subspace. It remains only to verify that individual plans are optimal. To this end, suppose that there is a trader i and consumption/income transfer plan (a^i, h^i) for trader i which belongs to the budget set at each node and has the property that $U^i(a^i) > U^i(x^i) + \delta$, for some $\delta > 0$. For each horizon T^*, we consider the consumption plan $a^i \mid T^*$ which coincides with a^i at each node s with $t(s) < T^*$, and is zero at every node s with $t(s) \geq T^*$. Recall that $\overline{U}_T^i$ is utility in the truncated economy. Continuity of preferences and the definition of the utility functions $\overline{U}_T^i$ guarantees that

$$\overline{U}_T^i(a^i \mid T^*) \geq U^i(a^i \mid T^*) > \overline{U}_T^i(x^i(T)) + \delta/2$$

for all $T > T^*$, provided that T^* is sufficiently large. Set $\underline{a}^i = (1-\epsilon)a^i \mid T^*$; continuity of preferences also guarantees that $U^i(\underline{a}^i) > U^i(x^i) + \delta/3$ for $\epsilon > 0$ sufficiently small. We set $\underline{h}^i = (1-\epsilon)h^i$. Because endowments are commensurable and the social endowment has value 1 at each node, individual wealths are bounded away from 0. Hence, the consumption/income transfer plan $(\underline{a}^i, \underline{h}^i)$ has the property that the budget and debt constraints are satisfied (for prices p and pricing functionals Q) with *strict* inequalities at every node.

For $T > T^*$ we define an income transfer plan $\underline{h}^i(T)$ by letting $\underline{h}_s^i(T)$ be the point of $K_s(T)$ closest to $\underline{h}_s^i$. Because $(\underline{a}^i, \underline{h}^i)$ satisfies the budget and debt constraints with strict inequalities at every node, convergence of income transfer subspaces $K_s(T) \to K_s$ and commodity spot prices $p_s(T) \to p_s$ implies that, for T sufficiently large, the consumption/income transfer plan $(\underline{a}^i, \underline{h}^i)$ strictly satisfies the budget and debt constraints (for prices $p(T)$ and pricing functionals $Q(T)$) at all nodes s with $t(s) \leq T^*$. Moreover, if T is large enough, the plan $(\underline{a}^i, \underline{h}^i)$ also strictly satisfies the implicit debt constraints $D_s^i(T)$ (for prices $p(T)$ and pricing functionals $Q(T)$) at every node s with $t(s) = T^*$.

The definition of the implicit debt constraints guarantees that it is therefore possible to find a consumption/income transfer plan (A^i, H^i) for the economy $\mathscr{E}(T)$ that agrees with $(\underline{a}^i, \underline{h}^i)$ for $t(s) < T^*$ and satisfies the budget constraints for the economy $\mathscr{E}(T)$ at *every* node. Since the consumption plan $\underline{a}^i$ is zero at every node s with $t(s) \geq T^*$, monotonicity of preferences means that $U^i(A^i) > U(\underline{a}^i)$. Hence, for T sufficiently large $\overline{U}_T^i(A^i) > \overline{U}^i(x^i(T)) + \delta/5$. Since (A^i, H^i) is feasible for the economy $\mathscr{E}(T)$, this is a contradiction. We conclude that the consumption/income transfer plans (x^i, h^i) are optimal, and hence that E^* is a pseudo-equilibrium, as desired.

It remains to see that the debt constraints D^i are loose and consistent. To this end, note first that our construction guarantees that the implicit debt constraints $D^i(T)$ are loose and consistent (with respect to the prices $p(T)$ and $Q(T)$) at every node s with $t(s) < T$. To see that the debt constraint D^i is loose, we fix a node s, an $\epsilon > 0$, and an income transfer plan $k_s^i \in K_s$ that satisfies the debt constraints at every node $\sigma \in s^+$, i.e. $k_s^i(\sigma) \geq D_\sigma^i$ for every $\sigma \in s^+$. Assumption 4 (Positive returns), together with the fact that all commodity spot prices are strictly positive and the fact that the range of the dividend operator $V_s(p)$ lies in the income transfer subspace K_s, implies that we can find an income transfer plan $h_s^i \in K_s$ that strictly satisfies the debt constraints at every node $\sigma \in s^+$ (i.e. $h_s^i(\sigma) > D_\sigma^i$ for every $\sigma \in s^+$) and which differs from k_s^i by at most ϵ at every node $\sigma \in s^+$. As before, we write $h_s^i(T)$ for the income transfer plan in the subspace $K_s(T)$ closest to h_s^i. Convergence of prices guarantees that, for $T > t(s)$ sufficiently large, the income transfer plan $h_s^i(T)$ satisfies the debt constraints at every node $\sigma \in s^+$, i.e. $h_s^i(T)(\sigma) \geq D_\sigma^i$ for every $\sigma \in s^+$. Because the debt constraints $D^i(T)$ are loose at s, it follows that

$$D_s^i(T) + p_s(T) \cdot w_s^i - Q_s(T) \cdot h_s^i(T) \leq 0.$$

Because $D_s^i(T) \to D_s^i$, $p_s(T) \to p_s$, $Q_s(T) \to Q_s$, and $h_s^i(T) \to h_s^i$, it follows that

$$D_s^i + p_s \cdot w_s^i - Q_s \cdot h_s^i \leq 0.$$

Since h_s^i differs from k_s^i by at most ϵ at every node $\sigma \in s^+$, and ϵ can be made as small as we like, we conclude that

$$D_s^i + p_s \cdot w_s^i - Q_s \cdot k_s^i \leq 0.$$

That is, the debt constraint D^i is loose at s. The argument that debt constraints are consistent is essentially the reverse of this argument; details are left to the reader. This completes the proof. □

The next lemma provides one of the crucial links between the various notions of a pseudo-equilibrium. It is important to recognize that it represents an assertion about *all* consistent systems of debt constraints, not just those that are compatible with some equilibrium.

Lemma C. In the presence of Assumptions 1–4, every consistent system of debt constraints that is uniformly bounded below is also bounded below by the finitely effective debt constraints.

Proof. We show that the debt constraints can almost be repaid in finite time. We fix commodity spot prices p and pricing functionals Q, let D^i be a loose, consistent system of debt constraints for trader i (relative to these prices), and let $\Delta < 0$ be a uniform lower bound for D^i (that is, $D^i_s \geq \Delta$ for each node s). For each node s, consistency of the debt constraint at s means the amount of debt D^i_s at node s can be repaid while meeting the debt constraints at succeeding nodes. That is, there is an income transfer plan k^i_s such that

- $p_s \cdot w^i_s + D^i_s = Q_s \cdot k^i_s$; and
- $k^i_s(\sigma) \geq D^i_\sigma$ for each node $\sigma \in s^+$.

We now fix a node s_0 and an amount of debt d with $0 > d > D^i_{s_0}$. The income transfer plan $k^i_{s_0}$ repays the debt $D^i_{s_0}$; to repay the amount d requires at most the fraction $\beta(s_0)$ of the income transfer plan $k^i_{s_0}$, where $\beta(s_0)$ solves the equation

$$p_{s_0} \cdot w^i_{s_0} + d = Q_{s_0} \cdot \left(\beta(s_0) k^i_{s_0} \right).$$

Solving for $\beta(s_0)$ yields:

$$\beta(s_0) = \frac{p_{s_0} \cdot w^i_{s_0} + d}{p_{s_0} \cdot w^i_{s_0} + D^i_{s_0}}.$$

Since the income transfer plan $k^i_{s_0}$ meets the debt constraints at each node $s_1 \in s_0^+$, the income transfer plan $\beta(s_0) k^i_{s_0}$ satisfies

$$\beta(s_0) k^i_{s_0}(s_1) \geq \beta(s_0) D^i_{s_1}$$

for each $s_1 \in s_0^+$ (keep in mind that $D^i_{s_1}$ is negative).

Now, at each node $s_1 \in s_0^+$, we can repeat the same process, using the new debt limit $\beta(s_0) D^i_{s_1}$. That is, we choose $\beta(s_1)$ to solve

$$p_{s_1} \cdot w^i_{s_1} + \beta(s_0) D^i_{s_1} = Q_{s_1} \cdot \left(\beta(s_1) k^i_{s_1} \right).$$

For each $s_2 \in s_1^+$, we then choose $\beta(s_2)$ to solve

$$p_{s_2} \cdot w^i_{s_2} + \beta(s_1) D^i_{s_2} = Q_{s_2} \cdot \left(\beta(s_2) k^i_{s_2} \right).$$

Continuing in this recursive fashion, we construct the coefficients $\beta(s_{t+1})$ to solve

$$p_{s_{t+1}} \cdot w^i_{s_{t+1}} + \beta(s_t) D^i_{s_{t+1}} = Q_{s_{t+1}} \cdot \left(\beta(s_{t+1}) k^i_{s_{t+1}} \right).$$

Recall our normalization: the value of the social endowment at each node is 1. Since endowments are commensurable, it follows that there is a $g > 0$ such that trader i's wealth at each node is at least g. We choose an integer T such that

$$g[1 - \beta(0)]T + \beta(0)\Delta \geq -g.$$

We claim that debt d at node s_0 can be repaid in $T+1$ periods. To see this, it suffices to consider any sequence $s_0, s_1, \dots, s_T$ (where $s_t \in s_{t-1}^+$ for each $t > 0$), and the associated income transfer plans $\beta(s_t)k_{s_t}^i$ constructed above, and show that $\beta(s_T)k_{s_T}^i$ leaves the next-period debt so small that it can be repaid from trader i's endowment.

To this end, we solve for $\beta(s_t)$ in terms of $\beta(s_{t-1})$, and recall that wealth is bounded below by $g > 0$ and debt constraints are bounded above by $\Delta < 0$ to conclude

$$\beta(s_t) = \frac{p_{s_t} \cdot w_{s_t}^i + \beta(s_{t-1}) D_{s_t}^i}{p_{s_t} \cdot w_{s_t}^i + D_{s_t}^i} \le \frac{g + \beta(s_{t-1})\Delta}{g + \Delta}. \tag{1}$$

Note in particular that $\beta(s_t) \le \beta(s_{t-1})$, so that $\beta(s_t) \le \beta(s_0)$ for all t. Solving Eq. (1) for $\beta(s_t)\Delta$ and using this fact yields:

$$\begin{aligned} \beta(s_t)\Delta &\ge g[1 - \beta(s_t)] + \beta(s_{t-1})\Delta \\ &\ge g[1 - \beta(s_0)] + \beta(s_{t-1})\Delta. \end{aligned} \tag{2}$$

Substituting $t = T$ in Eq. (2) and recalling the definition of T yields:

$$\beta(s_T)\Delta \ge g[1 - \beta(0)]T + \beta(0)\Delta \ge -g.$$

By construction, the debt that remains at any node $\sigma \in s_T^+$ is $\beta(s_T)k_{s_T}^i(\sigma) \ge \beta(s_T)\Delta$, so this debt can be repaid from the endowment at σ, and the debt at any node following σ will be zero, as desired.

We conclude that any amount of debt $d > D_{s_0}^i$ can be repaid in finite time. The definition of the finitely effective debt constraint FE_s^i as an infimum implies that $D^i \ge FE_s^i$; since s is arbitrary, this concludes the proof. □

The following lemma establishes the desired connection between a pseudo-equilibrium relative to a loose, consistent system of debt constraints and a finitely effective equilibrium.

Lemma D. In the presence of Assumptions 1–5, *if* $\langle p, K, Q, (x^i), (k^i) \rangle$ *is a pseudo-equilibrium relative to the loose consistent debt constraints* (D^i), *then these debt constraints* (D^i) *are necessarily uniformly bounded below and coincide with the finitely effective debt constraints* (FE^i). *In particular,* $\langle p, K, Q, (x^i), (k^i) \rangle$ *is a finitely effective equilibrium.*

Proof. To see that the debt constraints (D^i) are uniformly bounded below, we argue as in the proof of Lemma B. For each trader j, Lemma A implies that, for each node s, there are real numbers c, r, and δ, with $c > 0$, $0 < r < 1$, and $0 < \delta < 1$, such that the consumption plan

$$z_s^j(\delta) = (1 - \delta)x^j + \delta\langle x^j, c_s, rw^j \mid s \rangle$$

is preferred to the (pseudo-)equilibrium consumption plan x^j. Assumption 5 provides a uniform version of the conclusion of Lemma A: the numbers c and r may be chosen independently of node s (and independently of trader j, since the number of traders is finite).

We now suppose that the debt constraints D^i for trader i are not uniformly bounded below. Consider his actual equilibrium debts, i.e. $k^i_{s-1}(s)$. These actual debts are either unbounded below or bounded below. In the first case, we may argue just as in Lemma B. When $k^i_{s-1}(s)$ is very negative, trader i holds a large debt at the beginning of node s, so some other trader j must hold a large credit. However, then trader j could afford $z^j_s(\delta)$, which is preferred to his equilibrium consumption. This is a contradiction, so we conclude that trader i's actual equilibrium debts cannot be unbounded below. Then the differences between trader i's actual equilibrium debts and his debt constraints – i.e. the quantities $k^i_{s-1}(s) - D^i_s$ – must be unbounded above. Since trader i could repay the debt D^i_s while satisfying the next period's debt constraints, he could also certainly adopt a plan that would repay any fraction $\epsilon > 0$ of this debt. By adopting such a plan, trader i would free up resources at node s equal to $\epsilon(k^i_{s-1}(s) - D^i_s)$. Since these quantities are unbounded above, we conclude that, for some s, trader i would be able to afford consumption $z^i_s(\delta)$, which is preferred to his equilibrium consumption. This is a contradiction, so we conclude that trader i's actual equilibrium debts cannot be bounded below, either. Hence, trader i's debt constraints must be uniformly bounded below, as desired.

In view of Lemma C, the consistent debt constraints D^i are bounded below by the finitely effective debt constraints FE^i. Since the debt constraints D^i are also loose, and the finitely effective debt constraints are the suprema of all loose, consistent debt constraints, we conclude that $D^i = FE^i$ for each i, so the proof is complete. □

With these results in hand, the proofs of the theorems are quite simple.

Proof of Theorem 1. Lemma B guarantees the existence of a pseudo-equilibrium relative to some system of loose, consistent debt constraints; Lemma D guarantees that such a pseudo-equilibrium is necessarily finitely effective. □

Proof of Theorem 2. As we have already noted, the finitely effective debt constraints are loose and consistent. Hence a finitely effective pseudo-equilibrium is trivially a pseudo-equilibrium relative to a system of loose, consistent debt constraints. Of course, Lemma D yields the reverse.

Let us consider any finitely effective pseudo-equilibrium $E = \langle p, K, Q, (x^i), (k^i)\rangle$; we show that E is a bounded debt pseudo-equilibrium. Note first that Lemma C guarantees that the finitely effective debt constraints are bounded below; in particular, the consumption/income transfer plans (x^i, k^i) belong to the bounded debt budget sets. It remains to verify that the plans (x^i, k^i) are optimal among all plans with bounded debt. To this end, we fix a trader i and consider a plan $(\bar{x}^i, \bar{k}^i)$ for which the debts $\bar{k}^i_s$ are uniformly bounded below. For each node s, we write $l^i_s = \min\{\bar{k}^i, 0\}$. It is evident that l^i constitutes a consistent system of debt constraints, so Lemma C guarantees that $l^i_s \geq FE^i_s$ for each s, whence $\bar{k}^i \geq FE^i_s$ for each s. That is to say, the debts $\bar{k}^i$ satisfy the finitely effective constraints, so the plan $(\bar{x}^i, \bar{k}^i)$ belongs to the finitely effective budget set for trader i. Since the plan (x^i, k^i) is optimal in the finitely effective budget set, it follows that the consumption plan $\bar{x}^i$ is not strictly preferred to the consumption plan x^i. Hence E is indeed a bounded debt pseudo-equilibrium, as desired.

Finally, let $E = \langle p, K, Q, (x^i), (k^i)\rangle$ be a bounded debt pseudo-equilibrium. Since the actual debts are bounded, we may argue exactly as above to conclude that they satisfy the finitely effective constraints; hence the plans (x^i, k^i) belong to the finitely effective budget sets. To see that they are optimal, suppose not. Then there is a trader i and a consumption plan $\bar{x}^i$ that is strictly preferred to x^i and is financed by an income transfer plan that satisfies the finitely effective debt constraints. For each real number $\rho < 1$ and each date T, we define a consumption plan $\bar{x}^i(\rho, T)$ by

$$\bar{x}^i(\rho, T)_s = \begin{cases} \rho \bar{x}^i_s, & \text{if } t(s) < T, \\ 0, & \text{otherwise.} \end{cases}$$

Continuity of preferences guarantees that we can choose ρ sufficiently close to 1 and T sufficiently large that $\bar{x}^i(\rho, T)$ is strictly preferred to x^i. As in the proof of Lemma B, we see that $\bar{x}^i(\rho, T)$ can be financed by an income transfer plan that *strictly* satisfies the finitely effective debt constraints at each node s with $t(s) \leq T$. By definition, the debt at date T nodes can be repaid in finite time. Since $\bar{x}^i(\rho, T)$ involves no consumption after date $T - 1$, we can find an income transfer plan $\bar{k}^i$ that finances $\bar{x}(\rho, T)$ and leaves no debt from some finite time onward. Clearly such a plan has bounded debt, so $(\bar{x}^i, \bar{k}^i)$ belongs to the bounded debt budget set, and the consumption $\bar{x}^i$ is strictly preferred to x^i; this contradicts the optimality of the plan (x^i, k^i) in the bounded debt budget set. We conclude that (x^i, k^i) is indeed optimal in the finitely effective budget set, so E is a finitely effective equilibrium, as desired. This completes the proof. □

5. More general notions

In the preceding sections we explored several notions of an equilibrium which are defined by debt constraints. However, there are notions of an equilibrium – such as those of Hernandez and Santos, Magill and Quinzii, and Santos and Woodford – that do not fit easily into the framework we have developed. In this section we present a more general notion of an equilibrium which includes these notions as well as ours, and establish the equivalence of all equilibria satisfying this more general notion. [17]

As we have noted before, debt constraints, together with the budget balancing inequalities at each node, serve to define intertemporal budget sets (that is, the set of admissible plans of consumption and income transfers); different debt constraints define different budget sets. Here we consider *direct* restrictions on budget sets and the associated notion of a (pseudo-) equilibrium.

To formalize the pseudo-equilibrium notion we use, we fix an economy (as in Section 2), spot prices p, income transfer subspaces K, and pricing functionals Q. As before, we define (for each trader i) the pre-budget set $B^i(w^i, p, K, Q)$ to be the collection of consumption/income transfer plans (x^i, k^i) that satisfy the budget constraint at each node. For each i, let $F^i \subset B^i(w^i, p, K, Q)$ be a collection of consumption/income transfer plans. (At this point, we allow F^i to be arbitrary.) A tuple

$$\langle p, K, Q, (x^i), (k^i) \rangle$$

is a *pseudo-equilibrium relative to* $\{F^i\}$ if

- consumption plans are socially feasible;
- income transfer plans are socially feasible (i.e. $\sum_i k^i_s = 0$ for each s);
- for each trader i, the plan (x^i, k^i) maximizes trader i's utility over all plans belonging to F^i; and
- for each s, the range of the dividend operator $V_s(p)$ is a subspace of K_s.

This notion of an equilibrium is very general, allowing for entirely arbitrary budget sets F^i; of course, we wish to impose some additional discipline. To this end, we say that F^i *allows spot market trading* if:

(i) $(w^i, 0) \in F^i$, and

(ii) if $(x^i, k^i) \in F^i$ and $(\hat{x}^i, \hat{k}^i)$ is a plan such that $p_s \cdot x^i_s \geq p_s \cdot \hat{x}^i_s$ and $k^i_s = \hat{k}^i_s$ for each node s (so that the latter plan calls for the same income transfers but smaller expenditures on consumption), then $(\hat{x}^i, \hat{k}^i) \in F^i$.

Note that the pre-budget sets themselves satisfy these conditions, as do the bounded debt budget sets, and the budget sets defined by *any* system of debt constraints.

How much borrowing does a given budget set F^i allow? Following the reasoning of Section 2, we would insist that it should always be possible to acquire debt, provided that it can be repaid satisfying future constraints. To express that idea in the present context, consider consumption/income transfer plans α and β and a date T. Construct a plan γ by following the plan α at all nodes s (strictly) before date T and the plan β at all nodes on or after date T. We shall say that F^i

[17] For the case of financial assets, Florenzano and Gourdel (1993) have established equivalence of the equilibrium notions of Hernandez and Santos, Magill and Quinzii, and an equilibrium with loose consistent debt constraints.

is *loose* if the plan γ belongs to F^i whenever β belongs to F^i and γ belongs to the pre-budget set B^i. That is, F^i is loose if plans in the pre-budget set that meet the constraints implied by F^i from date 1 on necessarily belong to F^i (and hence meet the constraints implied by F^i at date 0), and plans in the pre-budget set that meet the constraints implied by F^i from date 2 on necessarily belong to F^i, and so forth.

It is easily seen that the budget sets associated to a system of debt constraints D^i are loose in this sense exactly when the debt constraints D^i are loose in the sense we have used previously. Thus, looseness of budget sets is the exact generalization of looseness of debt constraints. [18] In particular, the finitely effective budget sets are loose (and allow spot market trading).

Given an arbitrary budget set F, we may ask whether it is loose and allows spot market trading. If it is not, then we may enlarge it in an obvious way to obtain the smallest budget set $\hat{F}$ that is loose and allows spot market trading and contains F. (F can be expanded by adjoining all plans that call for the same income transfers but smaller expenditures on consumption as a plan in F, and all plans that meet the constraints implied by F from some time T onward; we call the resulting budget set F_1. Then F_1 can be expanded by adjoining all plans that call for the same income transfers but smaller expenditures on consumption as a plan in F_1, and all plans that meet the constraints implied by F_1 from some time T onward, etc. Continuing in this way – transfinitely if necessary – yields $\hat{F}$.) For instance, if we begin with the trivial budget set $F^i = \{(w^i, 0)\}$ and then take the closure (in the topology of convergence at each node) of $\hat{F}^i$, then we obtain exactly the finitely effective budget set. [19] In particular, every closed budget set that is loose and allows spot market trading contains the finitely effective budget set.

The following result generalizes Theorem 2 to this more general setting.

Theorem 3. In the presence of Assumptions 1–5 *the notions of a finitely effective pseudo-equilibrium and of a pseudo-equilibrium relative to some budget sets that are loose and allow spot market trading are equivalent. Indeed, if* $\langle p, K, Q, (x^i), (k^i)\rangle$ *is a pseudo-equilibrium relative to some budget sets* F^i *that are loose and allow spot market trading, then the budget sets* F^i *in fact coincide with the finitely effective budget sets* $B^i(w^i, p, K, Q, FE^i)$ *and* $\langle p, K, Q, (x^i), (k^i)\rangle$ *is a finitely effective pseudo-equilibrium.*

[18] Recall that every system of debt constraints is equivalent to a consistent system, in the sense of defining the same budget sets. For the same reason, it is unnecessary to impose a consistency requirement on budget sets.

[19] Note that taking the closure of the expanded budget set is relatively innocuous: if a plan (x^i, k^i) is optimal in the budget set $\hat{F}^i$, then it is also optimal in the closure of $\hat{F}^i$.

The proof is a straightforward modification of the proof ofTheorem 2, and is left to the reader.

It is easy to see that the pseudo-equilibrium notions of Hernandez and Santos, Magill and Quinzii, and Santos and Woodford are all pseudo-equilibria relative to appropriate budget sets, and that these budget sets are loose and allow for spot market trading. By Theorem 3, therefore, these pseudo-equilibrium notions coincide with a finitely effective equilibrium.

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