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DETERMINACY OF EQUILIBRIUM IN LARGE-SQUARE ECONOMIES*

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ABSTRACT

In this paper we argue that indeterminacy of equilibrium is a possibility inherent in economies with a double infinity of agents and goods, large-square economies. We develop a framework that is quite different from the overlapping generations one and that is amenable to analysis by means of differential calculus in linear spaces. The commodity space is a separable, infinite-dimensional Hilbert space, and each of a continuum of consumers is described by means of an individual excess demand function defined on an open set of prices. In this setting we prove an analog of the Sonnenschein-Mantel-Debreu theorem. Using this result, we show that the set of economies whose equilibrium sets contain manifolds of arbitrary dimension is non-empty and open in the appropriate topology. In contrast, we also show that, if the space of consumers is sufficiently small, then local uniqueness is a generic property of economies. The concept of sufficiently small has a simple mathematical formulation (the derivatives of individual excess demands form a uniformly integrable family) and an equally simple economic interpretation (the variations across consumers is not too great).

1. Introduction

Following Ostroy (1984), we call economies with a double infinity of agents and goods "large-square." An economy is determinate if equilibria are locally unique and indeterminate if there are a continua of equilibria. This paper studies the determinacy of equilibrium in large-square economies.

In the special case of overlapping generations, Kehoe and Levine (1985) have previously found robust examples of both indeterminate and determinate economies. This is in contrast to the well known result, due to Debreu (1970), that with finitely many consumers and commodities "most" economies are determinate. An account of the available theory may be found in Mas-Colell (1985) and Kehoe (1988). In this paper we argue that indeterminacy is not specific to the overlapping generations model, but is a possibility inherent in large-square economies.

We develop a framework that is quite different from the overlapping generations one and that is amenable to analysis by means of differential calculus in linear spaces. In our setting, the commodity space is a separable, infinite-dimensional Hilbert space, and each of a continuum of consumers is described by means of an individual excess demand function defined on an open set of prices. The advantage of Hilbert space is that we can use calculus. The disadvantage is that the use of calculus necessitates that the price domain (and, implicitly, the consumption set) has a non-empty interior. This means that we are allowing, to some extent, negative prices and consumption. Consequently, our economies have a different flavor than in the standard case.

It might be thought, therefore, that the source of indeterminacy is that we allow a larger price space than usual. Indeed, in our infinite dimensional setting, we cannot rule out the possibility that robust indeterminacy requires negative prices. This concern is mitigated, however, by the observation that in our setting a finite dimensional commodity space implies generic determinacy even allowing negative prices.

The use of Hilbert space has other implications: the value of the aggregate endowment is necessarily finite in any equilibrium. Consequently, every equilibrium is Pareto efficient and fiat money has no role to play. Since indeterminacy is possible, however, this reinforces the view that indeterminacy does not depend on the possibility of Pareto inefficiency or on fiat money.

We introduce a topology on the space of economies that is a natural generalization of the usual topology in the finite-dimensional setting. In the latter the set of determinate economies contains an open, dense set. By contrast, we show in Section 5, and this our first conclusion, that in our more general setting there are open sets of economies whose equilibrium sets contain manifolds of arbitrary dimension. Thus, we obtain robust examples of indeterminacy.

To attain this result, we show that an analog of the Sonnenschein-Mantel-Debreu theorem is true; that is, any continuous, bounded function that satisfies Walras's law and is homogeneous of degree zero can arise as the (mean) excess demand function of an economy. This is a substantial extension of the finite-dimensional result because of some subtle convergence issues. With this theorem, we can obtain economies for which the manifold of equilibrium prices has any prescribed dimension. Recall a basic difference between finite-dimensional and infinite-dimensional spaces: in a finite-dimensional space every linear operator that is onto necessarily has a trivial kernel, while in an infinite-dimensional space there are linear operators that are onto but that have kernels of arbitrary dimension. We can use any one of these linear operators to generate an example of indeterminacy that, by the implicit function theorem, is robust. The idea is already implicit in Samuelson's (1958) paper on overlapping generations: with infinitely many equations and unknowns it is hard to meaningfully say that they are equal in number.

In Section 6 we look at the other side of the coin, that is, at determinacy, and obtain our second class of results. From the work of Muller and Woodford (1984) and of Kehoe, Levine, and Romer (1987) on economies with finite number of (types of) consumers, one is naturally led to suppose that, if the space of consumers is sufficiently small, then local uniqueness should be a generic property. We show that this supposition is correct. The precise expression of "sufficiently small" is a condition we call C^1 integrability. It has a simple mathematical formulation (the derivatives of individual excess demands form a uniformly integrable family) and an equally simple economic interpretation (the variation of tastes across consumers is not too great). The condition is automatically satisfied if there are only finitely many (types of) consumers, or only finitely many commodities. In the general large-square framework, the class of economies satisfying this condition is open (but, of course, not dense) in the class of all economies. Within this class the determinate economies are dense. The key fact is that, roughly, C^1-integrable economies are like finite dimensional ones in that the derivative of mean excess demand has a trivial kernel whenever it is onto. The Slutsky decomposition of individual excess demand derivatives plays a key role here, as does the mathematical theory of semi-Fredholm operators.

2. The Model of an Economy

2.A. The Commodity Space

Our commodity space is an arbitrary separable infinite-dimensional Hilbert space H. The inner (dot) product of two vectors h, h' in H is denoted by $h \cdot h'$, and the norm of the vector h by $\|h\| = (h \cdot h)^{1/2}$.

The restriction to Hilbert space is a limitation, but it facilitates the application of differential calculus and differential topology. At the same time, it is general enough to accommodate the points we wish to make. As is well known, it is often possible, by changing units, to bring into our Hilbert space framework economic situations that are not naturally described in those terms. For general facts about Hilbert space, we refer to Dunford and Schwartz (1958).

2.B. Prices

Fix $e \in H$ with $\|e\| = 1$. The normalized set of prices is a fixed, bounded, convex, relatively open set $\hat{U} \subset \{p: p \in H, p \cdot e = 1\}$ such that $e \in \hat{U}$. Our set of unnormalized prices is $U = \{\lambda p: p \in \hat{U}, \delta < \lambda < 1/\delta\}$ where $\delta > 0$ is a fixed number. Note that $\delta < p \cdot e < 1/\delta$ for each $p \in U$, and that $\sup_{p \in U} \|p\| < \infty$.

Two comments are in order here: First, we have not assumed that the space H is equipped with an order structure, so that it is meaningless to speak of positive prices. Even if we were to assume an order structure on H, however, it could not be the case that the price domain would be contained in the positive cone of H since the latter has empty interior. Thus, we are definitely allowing for some non-positive prices. Since the view taken in this paper is that the determinacy problem is not conceptually related to positivity of prices, it is natural to follow the most convenient route and assume an open price domain. The second comment is that we are not considering prices (and, therefore, neglecting the possibility of equilibrium prices) belonging to spaces larger than H. For our results on indeterminacy (Section 5) this is no limitation, but it could conceivably be for our results on determinacy (Section 6).

2.C. Consumers

We specify consumers as excess demand functions.

Definition 2.1. A function $\zeta\colon\ U \to H$ is an individual excess demand function if:

(i) (Differentiability) $\zeta\colon\ U \to H$ is a C^1 function; that is, ζ is continuously differentiable;

(ii) (Boundedness) $\sup_{p \in U} \|\zeta(p)\| < \infty$;

(iii) (Walras's law) For all p in U, $p \cdot \zeta(p) = 0$;

(iv) (Degree zero homogeneity) If $\lambda > 0$ then $\zeta(\lambda p) = \zeta(p)$ for each p, λp in U;

(v) (Weak axiom of revealed preference) If $p \cdot \zeta(p') \leq 0$ and $p' \cdot \zeta(p) \leq 0$ then $\zeta(p) = \zeta(p')$.

Assumption (i) has the same interpretation as in the finite-dimensional case; see Lang (1962) for an introduction to calculus in infinite-dimensional spaces. Because H is a Hilbert space, there are many C^1 excess demand functions that arise from preference maximization; see Araujo (1985). Assumption (ii) is made only for convenience, but it is automatically satisfied if the price domain U is not too large. Assumptions (iii) and (iv) need no comment. (In fact, (iv) is redundant, since it is implied by (iii) and (v).) In Assumption (v), we require the weak axiom rather than preference maximization because that is all we need for our positive results in Section 5. The weak axiom is important because of the following familiar implications.

Proposition 2.1. *If* $\zeta\colon U \to H$ *is an individual excess demand function, then for all* p *in* U, *the linear operator* $D\zeta(p)\colon H \to H$ *is negative quasi-semi-definite on* $\{h\colon h \cdot \zeta(p) = 0\}$. *That is,* $h \cdot D\zeta(p)h \le 0$ *whenever* $h \cdot \zeta(p) = 0$.

Proof. The proof is exactly the same as in the finite-dimensional case; see, for example, Mas-Colell (1985; 5.7.3 and 2.4.3). ∎

In our Hilbert space framework, the combination of Walras's law and the weak axiom has strong implications. Suppose, for example, we take H to be the space ℓ_2 of square-summable sequences. Then there is an integer N such that any individual excess demand function with $\zeta_i(p) = 0$ for $1 \le i \le N$ is identically zero on an open subset of U. That is, if the demand for the first N commodities is identically zero, then demand for all commodities is identically zero. To see this, observe that if N is large enough, then there is a point of the form $(p_1, p_2, \ldots, p_N, 0, \ldots) \in U$. Since U is open, it follows that there is an open subset $B \subset U$ so that, if $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_N, 0, \ldots)$, $p = (0, \ldots, p_{N+1}, p_{N+2}, \ldots)$, and $\bar{p} + p \in B$, so is $\bar{p} - p \in B$. Fix such a $\bar{p}$ and p. Since $\zeta_i = 0$ for $i = 1, 2, \ldots, N$, $\bar{p} \cdot \zeta(\bar{p}+p) = 0$, so the weak axiom implies either $p \cdot \zeta(\bar{p}) = (\bar{p}+p) \cdot \zeta(\bar{p}) > 0$ or $\zeta(\bar{p}+p) = \zeta(\bar{p})$; similarly, since $\bar{p} \cdot \zeta(\bar{p}-p) = 0$, either $-p \cdot \zeta(\bar{p}) > 0$ or $\zeta(\bar{p}-p) = \zeta(\bar{p})$. Since $p \cdot \zeta(\bar{p}) > 0$ and $-p \cdot \zeta(\bar{p}) > 0$ are not both true, either $\zeta(\bar{p}+p) = \zeta(\bar{p})$ or $\zeta(\bar{p}-p) = \zeta(\bar{p})$. Finally, $0 = (\bar{p}+p) \cdot \zeta(\bar{p}+p) = p \cdot \zeta(\bar{p}+p)$ implies $0 = -p \cdot \zeta(\bar{p}+p) = (\bar{p}-p) \cdot \zeta(\bar{p}+p)$. Similarly, $(\bar{p}+p) \cdot \zeta(\bar{p}-p) = 0$.

Consequently, the weak axiom implies $\zeta(\bar{p}+p) = \zeta(\bar{p}-p)$. We conclude that $\zeta(\bar{p}+p) = \zeta(\bar{p})$. Fix $M > N$, and set $p_i = 0$ for $i \neq M$. Walras's law implies $0 = p_M\zeta_M(\bar{p}+p) = p_M\zeta_M(\bar{p})$. Consequently, $\zeta_M(\bar{p}) = 0$ for any $M > N$, and we conclude that $\zeta(p) = 0$ for $p \in B$.

This remark shows, among other things, that the models we study do not include overlapping generations type models. Therefore, the instances of robust indeterminacy we construct in Section 5 are not merely thinly disguised versions of the familiar examples of indeterminacy in that model. They are, rather, independent specimens of the phenomenon. This lends support to one of our theses: that the possibility of robust indeterminacy is not specific to some double-infinity models, but rather is quite general.

2.D. <u>Integration and Mean Demand</u>

Our set of consumers is indexed by the interval $[0,1]$, equipped with Lebesgue measure. For any given price, demand is a function from $[0,1]$ to H. To define the concept of mean demand we need a notion of integration for such functions. In this section, we gather the required essentials on integration of Hilbert space valued functions.

<u>Definition 2.2</u> . The function $x: [0,1] \to H$ is <u>measurable</u> if there is a sequence of (measurable) simple functions $x_n: [0,1] \to H$ such that $\|x(t)-x_n(t)\| \to 0$ for almost every t in $[0,1]$. The measurable function x is <u>integrable</u> if there is a sequence $\{x_n\}$ of simple functions such that $\int\|x(t)-x_n(t)\|dt \to 0$, in which case

$$\int x(t)dt = \lim_{n\to\infty} \int x_n(t)dt.$$

(We usually write $\int x(t)dt$ instead of $\int_{[0,1]} x(t)dt$.)

A number of comments need to be made. The first is that, because we have assumed H to be separable, measurability of a function $x\colon [0,1] \to H$ is equivalent to the requirement that $x^{-1}(W)$ be measurable for each open subset W of H. The second is that, as it should be, if x is integrable, then $\int\|x(t)-y_n(t)\|dt \to 0$ for every sequence of simple functions (y_n) with $\|x(t)-y_n(t)\| \to 0$ for almost every t. As a consequence, $\lim_{n\to\infty}\int y_n(t)dt = \int x(t)dt$ for any such sequence (y_n). Finally, if $x\colon [0,1] \to H$ is measurable, so is $\|x(\cdot)\|\colon[0,1] \to \mathbb{R}$, and x is integrable if and only if $\int\|x(t)\|dt < \infty$. This type of integral is usually called the Bochner integral. It is the most natural infinite-dimensional generalization of the (scalar) Lebesgue integral, and has most of its good properties. For example, the dominated convergence theorem holds. We refer the reader to Diestel and Uhl (1977) for more information.

2.E. Economies, Mean Excess Demand Functions, and Equilibrium

Definition 2.3 An economy is a function $z\colon U \times [0,1] \to H$ such that:

(i) for each t, $z_t = z(\cdot,t)\colon U \to H$ is an individual excess demand function;

(ii) for each p in U, $z(p,\cdot)\colon [0,1] \to H$ is an integrable function.

(iii) $\sup_{p\in U} \|z(p,\cdot)\|$ is integrable.

Although no joint measurability of requirement has been imposed on the mapping $z\colon U \times [0,1] \to H$, it is not difficult to show that joint measurability follows from the conditions we have imposed. Since we have no use for this fact, however, we omit the proof. Further comments on conditions (i) and (ii) can be found below in Subsections 2.C and 2.D.

<u>Definition 2.4</u>. The <u>mean excess demand</u> $Z\colon U \to H$ of an economy z is defined by

$$Z(p) = \int z_t(p)dt.$$

<u>Proposition 2.2.</u> *Mean excess demand is continuous.*

<u>Proof</u>. Fix a price q and a sequence of prices $q_n \to q$. Then $z_t(q_n) \to z_t(q)$ for each t; moreover by (iii), $\|z_t(q_n)\|$ is dominated by an integrable function of t, and the Lebesgue dominated convergence theorem applies to show that:

$$Z(q) = \int z_t(q)dt = \int \lim_{n\to\infty} z_t(q_n)dt = \lim_{n\to\infty} \int z_t(q_n)dt = \lim_{n\to\infty} Z(q_n).$$

Of course, this says that Z is continuous at q as desired. ∎

The restriction to a finite mass of consumers is not a severe limitation since what really matters is aggregate (or mean) demand. It is always possible to rescale so that the mass of consumers becomes finite while aggregate demand remains unchanged. For example, given countably many (types of) consumers with bounded excess demand functions $x_1, x_2, \ldots,$ and such that $\Sigma_{i=1}^{\infty} \|x_i(p)\| < \infty$ for each p, we may replace the ith consumer (or type) by the interval $(2^{-i}, 2^{-i+1}]$ and scale the excess demand function so that $z_t(p) = 2^i x_i(p)$ for t in $(2^{-i}, 2^{-i+1}]$. In the rescaled economy, mean demand is

$$Z(p) = \int z_t(p)dt = \sum_{i=1}^{\infty} \int_{(2^{-1}, 2^{-i+1}]} 2^i x_i(p)dt = \sum_{i=1}^{\infty} x_i(p).$$

With the hypotheses made to this point, mean excess demand need not be C^1. It is C^1 if, for example, there is a suitably uniform bound on derivatives of individual excess demand. As we shall see in Section 6, however, this is far from being an innocuous technical restriction. Indeed, the existence of such a bound has strong implications. We prefer, therefore, to impose differentiability of mean excess demand as a matter of definition.

Definition 2.5. The economy z has smooth demand (or is a smooth demand economy) *if the mean excess demand function* Z *is* C^1.

The following definition needs no comment.

Definition 2.6. The price vector p in U is an equilibrium price for the economy z if $Z(p) = 0$.

2.F. The Space of Economies

We denote the set of economies by $\mathcal{E}$. For our purposes, the distance between two economies should reflect not only the difference between their levels of excess demand, but also the difference between the derivatives of excess demand. That is, two economies z, z' should be close if $z_t(p)$ and $z'_t(p)$ are close on average and $Dz_t(p)$ and $Dz'_t(p)$ are close on average. There is a natural way to make this informal idea precise.

Definition 2.7. For z, z' in $\mathcal{E}$

$$d(z,z') = \min\left\{1, \int\left[\sup_{p\in U} \|z_t(p)-z'_t(p)\| + \sup_{p\in U} \|Dz_t(p)-Dz'_t(p)\|\right]dt\right\}$$

It is easily checked that this does indeed define a metric on $\mathcal{E}$. Lemma A.1 of the Appendix implies that the integrand is measurable.

Although this metric topology of $\mathcal{E}$ is badly disconnected, it has some good properties that make it just right for our objectives.

Proposition 2.3.

(i) *If* z *is a smooth demand economy and* z' *is an economy with* $d(z,z') < 1$, *then* z' *is also a smooth demand economy. In particular, the set of smooth demand economies is an open subset of* $\mathcal{E}$.

(ii) *If* $\{z^n\}$ *is a sequence of smooth demand economies that converges to the economy* z, *then* z *is a smooth demand economy and*

$$\sup_{p\in U} \|Z^n(p)-Z(p)\| + \sup_{p\in U} \|DZ^n(p)-DZ(p)\| \to 0.$$

In particular, the set of smooth demand economies is a closed subset of $\mathcal{E}$.

Proof.

(i) Write $x_t = z_t - z'_t$ and $X = Z - Z'$ so that

$$X(p) = Z(p)-Z'(p) = \int(z_t(p)-z'_t(p))dt = \int x_t(p)dt.$$

It suffices to show that X is C^1. To this end, fix p in U, q in H, and let λ be a non-zero real number sufficiently small that $p + \lambda q \in U$. Then

$$\frac{1}{\lambda}\Big(X(p+\lambda q) - X(p)\Big) = \int \frac{1}{\lambda}\Big(x_t(p+\lambda q) - x_t(p)-DZ(p)\Big)dt.$$

As $\lambda \to 0$, the integrand tends (pointwise) to $Dx_t(p)q$. Moreover, the mean value theorem (see Abraham and Robbin (1967)) implies that the norm of the integrand is dominated by

$$\|q\| \sup_{\bar{p}\in U} \|Dx_t(\bar{p})\| = \|q\| \sup_{\bar{p}\in U} \|Dz_t(\bar{p})-Dz'_t(p)\|,$$

which is, by assumption, an integrable function of t. The dominated convergence theorem then implies that

$$DX(p)q = \lim_{\lambda \to 0} \frac{1}{\lambda} (X(p+\lambda q) - X(p))$$

$$= \int \lim_{\lambda \to 0} (x_t(p+\lambda q) - x_t(p))dt$$

$$= \int Dx_t(p)q \, dt.$$

Moreover,

$$\|DX(p)q\| \le \int \|Dx_t(p)q\|dt$$

$$\le \int \|q\| \sup_{\bar{p} \in U} \|Dx_t(\bar{p})\|dt$$

$$\le \|q\|d(z,z').$$

This means that X is differentiable and that $\|DX(p)\| \le d(z,z') < 1$ for each p. Continuity of DX follows as in Proposition 2.2. This proves (i).

(ii) Since $z^n \to z$, there is an index n_0 such that $d(z^n,z) < 1$ for $n \ge n_0$. In particular, (i) implies that z is a smooth demand economy. In addition, for $n \ge n_0$ and p in U,

(2.1) $$\|Z^n(p) - Z(p)\| \le \int \|z_t^n(p) - z_t(p)\|dt.$$

Moreover, for each q in H,

$$\|DZ^n(p)q - DZ(p)q\| \le \int \|Dz_t^n(p)q - Dz_t(p)q\|dt.$$
$$\le \|q\| \int \|Dz^n(p) - Dz(p)\|dt,$$

so that

(2.2) $$\|DZ^n(p) - DZ(p)\| \le \int \|Dz^n(p) - Dz(0)\|dt.$$

If we take suprema over p in U and combine the inequalities (2.1) and (2.2), we obtain

$$\sup_{\bar{p} \in U} \|Z^n(\bar{p}) - DZ(p)\| + \sup_{\bar{p} \in U} \|DZ^n(\bar{p}) - DZ(\bar{p})\| \le d(z^n,z)$$

Since $d(z^n,z) \to 0$, this gives the desired result. ∎

3. Regular Equilibria

Consider a fixed smooth demand economy z with mean excess demand function Z. Define $T_p = \{h\colon\ h\cdot p=0\}$. Because Z is homogeneous of degree zero, $DZ(p)p = 0$ and, consequently, $DZ(p)(H) = DZ(p)(T_p)$. This implies that the kernel of $DZ(p)$, which contains p, has dimension at least one. On the other hand, differentiating Walras's law yields

$$p \cdot DZ(p)q + q \cdot Z(p) = 0$$

for p in U and q in H. Hence, $DZ(p)(H) \subset T_p$ if $Z(p) = 0$. Thus, at an equilibrium price p, $DZ(p)$ maps T_p into itself. The following definition of regular equilibrium is thus in complete analogy with the usual finite-dimensional theory. (See, for example, Mas-Colell (1985) and Kehoe (1988).)

Definition 3.1. Let z be a smooth demand economy. The equilibrium price p is *regular* if $DZ(p)$ maps T_p onto T_p. The economy is *regular* if every equilibrium price is regular.

As the following proposition shows, regular equilibria are well behaved: the set of nearby equilibrium prices is a manifold that depends continuously on the economy.

Proposition 3.1. *Let $\bar{p}$ be a regular equilibrium price of the smooth demand economy $\bar{z}$. Then there is an open set $U' \subset U$, a non-empty, relatively open set $V \subset$ kernel $D\bar{Z}(\bar{p})$, an open set $W \subset \mathcal{E}$, and a continuous map $\psi\colon V \times W \to U'$ such that:*

(i) $\bar{p} \in U'$, $\bar{z} \in W$, *and every z in W is a smooth demand economy;*

(ii) *If* z *is an economy in* W *with mean excess demand* Z *and* $p = \psi(v,z)$, *then* $Z(p) = 0$. *Conversely, if* $Z(p) = 0$ *for some* p *in* U', *then there is a unique* v *in* V *with* $\psi(v,z) = 0$.

(iii) *For every* z *in* W, $\psi(\cdot,z)\colon V \to U'$ *is a diffeomorphism onto*

$$\{p\colon\ p \in U' \text{ and } Z(p) = 0\}.$$

Proof. Notice first of all that, since $\bar{Z}$ is a C^1 function, there is an open set U_0 with $\bar{p} \in U_0$ such that $\|Z(p)\|$ and $\|DZ(p)\|$ are uniformly bounded on U_0. It is now convenient to make a normalization. By assumption, there is a vector e in H with $\|e\| = 1$ such that $p \cdot e > 0$ for all p in U. Set $\hat{U}_0 = \{p\colon\ p \in U_0, p \cdot e = 1\}$, and define $\hat{Z}\colon\ T_e \to T_e$ by

$$\hat{Z}(p) = Z(p) - e \cdot Z(p)e.$$

(Recall that $T_e = \{h\colon\ h \cdot e = 0\}$.) It is easily checked that $\hat{Z}(p) = 0$ if and only if $Z(p) = 0$ and that $D\hat{Z}(p)$ is onto T_e if p is a regular equilibrium price.

Let B denote the Banach space of functions $f\colon\ \hat{U}_0 \to T_e$ that are C^1, are bounded, and have a bounded derivative; the norm on B is

$$\|f\| = \sup_{p \in \hat{U}_0} \|f(p)\| + \sup_{p \in \hat{U}_0} \|Df(p)\|.$$

By Proposition 2.3, there is a neighborhood $\mathcal{E}'$ of $\bar{z}$ in $\mathcal{E}$ on which the map $x\colon\ \mathcal{E}' \to B$ given by

$$x(z) = \hat{Z} - \hat{\bar{Z}}$$

is well defined and continuous.

Set $E_0 = \text{kernel } D\hat{Z}(p) \subset T_e$, and let E_1 be the orthogonal complement of E_0 in T_e. (Keep in mind that T_e inherits a Hilbert space structure from H.) For each p in T_e, we can write p uniquely as $p = p_0 + p_1$ where $p_0 \in E_0$, $p_1 \in E_1$. We can find relatively open subsets E_0' of E_0 and E_1' of E_1 such that $\bar{p}_0 \in E_0'$, $\bar{p}_1 \in E_1'$ and $E_0' + E_1' \subset \bar{U}_0$.

We now define the function ψ: $E_0' \times E_1' \times B \rightarrow T_e$ by

$$\psi(p_0,p_1,G) = \hat{\bar{Z}}(p_0+p_1) + G(p_0+p_1) \ .$$

The function ψ is C^1 (see Abraham and Robbin (1967, p. 25)); moreover $\psi(\bar{p}_0,\bar{p}_1,0) = 0$ and $D_0\psi(\bar{p}_0,\bar{p}_1,0)$ is onto. (Here, $D_0\psi$ is the derivative of ψ considered only as a function of the E_0' variable; that is, $D_0\psi(\bar{p}_0,\bar{p}_1,0)$ is the derivative at $\bar{p}_0$ of the function $p_0 \rightarrow \psi(p_0,\bar{p}_1,0)$: $E_0' \rightarrow T_e$.) Hence the implicit function theorem (see Lang (1962)) provides an open neighborhood E_0'' of $\bar{p}_0$ in E_0', an open neighborhood E_1'' of $\bar{p}_1$ in E_1', an open neighborhood J of 0 in B, and a C^1 function ξ: $E_1'' \times J \rightarrow E_0''$ such that $\xi(\bar{p}_0,0) = \bar{p}_1$ and $\xi(p_0,G) = p_1$ if and only if

$$(p_0,p_1,G) \in E_1'' \times E_1'' \times J \quad \text{and} \quad \psi(p_0,p_1,G) = 0.$$

Now choose an open neighborhood W of $\bar{z}$ in $\mathcal{E}'$ such that $\hat{Z} \in J$ whenever $z \in W$. Set $V = E_0''$, $U' = E_0'' + E_1''$, and define ψ: $V \times W \rightarrow U'$ by

$$\psi(v,z) = v + \xi(v,\hat{Z}-\hat{\bar{Z}}).$$

It is straightforward to check that ψ has all the desired properties. ∎

We distinguish two types of regular equilibria. If p is a regular equilibrium and kernel $DZ(p)$ has dimension one, we say that p is <u>determinate</u>; if kernel $DZ(p)$ has dimension greater than one, we say that p is <u>indeterminate</u>. Recall that kernel $DZ(p)$ always has dimension at least one. In view of Proposition 3.1 and the homogeneity of mean demand, determinacy of a regular equilibrium price p means that the relative prices at

$\bar{p}$ are isolated in the equilibrium set. On the other hand, indeterminacy means that relative prices at $\bar{p}$ are not isolated in the equilibrium set. More generally, if the dimension of kernel $DZ(\bar{p})$ is $k \geq 1$, then, near $\bar{p}$, the set of relative equilibrium prices is a (k-1)-dimensional manifold. Notice that Proposition 3.1 also implies that determinacy and indeterminacy are both robust properties of regular equilibria.

We remark that determinacy does not mean that there is necessarily only a finite number of equilibrium relative prices. That equilibria are locally isolated only enables us to conclude that there are countably many equilibria. The problem stems from the non-compactness of the price space. In the finite case this type of problem can be rectified using an appropriate boundary condition, but in the infinite case no such solution is available.

If the commodity space H is finite-dimensional (that is, if there are only a finite number of commodities), then regularity of $\bar{p}$ means that $DZ(\bar{p})$ maps onto $T_{\bar{p}}$ and hence corank $DZ(\bar{p}) = 1$. Consequently, rank $DZ(\bar{p}) = \dim(\text{kernel } DZ(\bar{p})) = 1$. In other words, if H is finite-dimensional then all regular equilibria are determinate.

Our goal in the remaining sections is to show that, if H is infinite-dimensional, then

(i) there is a large class of economies with regular, indeterminate equilibria;

(ii) there is also a large class of economies for which all regular equilibria are determinate.

The dividing line between these two classes of economies is uniform integrability of the derivatives of individual excess demand. The class of economies possessing this uniform integrability properly includes those with a finite-dimensional commodity space and those with a finite number of (types of) consumers. It constitutes their natural generalization. Within this class of economies, regular equilibria are determinate and regular economies are dense. On the other side, the class of economies not possessing this uniform integrability property includes economies with regular equilibria that are not determinate as well as regular economies with regular equilibria that are determinate; we do not know whether regular economies are dense in this class.

4. The Class of Mean Excess Demand Functions

In this section, we show that the underlying economic structure places virtually no restriction on the form of the mean excess demand function; that is, we establish the analog of the Sonnenschein-Mantel-Debreu theorem (see, for example, Shafer and Sonnenschein (1981)). Consequences of this result are discussed in Section 5.

Theorem 4.1. *Let* $Z\colon U \to H$ *be a continuous, bounded function that is homogeneous of degree zero and satisfies Walras's law. Then* Z *is the mean excess demand function of some (integrably bounded) economy* z.

Before giving the proof, we make a comment. The argument we give is a variation on Debreu's. It is not, however, a straightforward variation. The rather subtle reason goes as follows: Given an orthonormal basis $\{b_i\}$ for H, every vector x in H admits a unique decomposition

$$x = \sum_{i=1}^{\infty} \lambda_i b_i,$$

and this (infinite) sum converges in the norm topology of H. It is not generally be true, however, that this series converges absolutely; that is, the sum $\Sigma\|\lambda_i b_i\|$ is not generally be finite because the sequence $\{\lambda_i\}$ of coefficients generally is square-summable but not summable. If we try to express a given function Z, which is homogeneous of degree zero and satisfies Walras's law, as an integral of individual excess demand functions by the familiar Debreu procedure, we are led to decompose $Z(p)$ in terms of an orthonormal basis for $T(p)$: $Z(p) = \Sigma\lambda_i(p)b_i(p)$. That this series may fail to converge absolutely will generate an integral expression for $Z(p)$ that is not convergent. Avoiding this difficulty requires some work.

Before turning to the proof proper, we discuss some preliminary constructions. Recall our standing assumption that there is a vector e in U with $\|e\| = 1$, and a positive number δ such that the price domain U has the form

$$U = \{\lambda\hat{p}: \hat{p} \in \hat{U}, \delta < \lambda < \frac{1}{\delta}\}$$

for some bounded, relatively open, convex subset $\hat{U}$ of $\{p: p\cdot e=1\}$. Write E_0 for the one-dimensional space spanned by e and E_1 for its orthogonal complement. Let Q be the orthogonal projection onto E_1, $Qx = x - (x\cdot e)e$. Then every x in H has a unique decomposition $x = \lambda(x)e + Qx$, where $\lambda(x) = x \cdot e$ is a continuous linear functional.

Since H is separable, we may choose, and fix, a countable dense subset $\{e_j\}$ of the unit sphere $S = \{x: x\in E_1, \|x\|=1\}$ of E_1. We let ℓ_1 denote the Banach space of all sequences $\alpha = (\alpha_n)$ that are summable, in other words, $\|\alpha\| = \Sigma|\alpha_n| < \infty$. Define a continuous linear operator $A: \ell_1 \to E_1$ by setting

$$A(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n e_n .$$

<u>Lemma 4.2</u>. *The operator* A *maps* ℓ_1 *onto* E_1.

Proof. That A is well defined and continuous is an easy consequence of the triangle inequality. To see that A is onto, let x be a vector in E_1; there is no loss of generality in assuming that $\|x\| = 1$. Since (e_n) is dense in the unit sphere S of E_1, we can find an index n_0 with $\|x-e_{n_0}\| < 1/2$; write $x_1 = x - e_{n_0}$. Next, we can find an index n_1 such that $\|x_1-\|x_1\|e_{n_1}\| < 1/4$; write $x_2 = x_1-\|x_1\|e_{n_1}$. Continuing in this way, we find a sequence (x_j) in E_1 and a sequence (n_j) of indices such that $\|x_j\| < 2^{-j}$ and $x_{j+1} = x_j-\|x_j\|e_{n_j}$. Then $x = e_{n_0} + \Sigma_{j=1}^{\infty} \|x_j\|e_{n_j}$, so that x is the image under A of the sequence (α_n) where $\alpha_0 = 1$ $\alpha_n = \|x_j\|$ if $n = n_j$, $\alpha_n = 0$ otherwise. Because $\|x_j\| = 2^{-j}$, the sequence (α_n) is in ℓ_1; in fact, $\|(\alpha_n)\| < 2$. ∎

We are now interested in finding a continuous function σ^*: $E_1 \to \ell_1$ such that $A(\sigma^*(x)) = x$ for all $x \in E_1$. The existence of a linear σ^* is not guaranteed. Bartle and Graves (1952) show, however, that a (possibly) non-linear σ^* does exist. This also follows from the more general selection theorem of Michael (1956). Michael's theorem says that there is a continuous selection from a lower hemi-continuous, closed, convex, non-empty valued correspondence. Here A^{-1} is a correspondence satisfying precisely those properties, and σ^* is the continuous selection.

Lemma 4.3. *There is a continuous (possibly non-linear) function* σ: $E_1 \to \ell_1$ *and a positive number* ρ *such that* $\sigma(0) = 0$, $A(\sigma(x)) = x$, *and* $\|\sigma(x)\| \leq \rho(1+\|x\|)$ *for each* x.

Proof: We may clearly assume that the selection σ^* above satisfies $\sigma^*(0) = 0$. Since σ^* is continuous, there is a ball (of radius $\mu > 0$, say) on which σ^* is bounded; say $\|\sigma^*(x)\| \leq r$ for $\|x\| = \mu$. Now define

σ: $E_1 \to \lambda_1$ by

$$\sigma(x) = \sigma^*(x) \quad \text{if} \quad \|x\| = \mu,$$

$$\sigma(x) = \frac{\|x\|}{\mu}\, \sigma^* \left(\frac{\mu}{\|x\|}\, x\right) \quad \text{if} \quad \|x\| > \mu .$$

Since A is linear, it is easy to see that σ has the required properties. ∎

Recall that Q is the orthogonal projection on E_1. The next lemma deals with a certain class of individual excess demand functions.

Lemma 4.4. *Let* y *be a vector in* E_1 *and let* c *be a positive real number. Then the function* $\zeta(y,c)$: U → H *given by*

$$\zeta(y,c)(p) = \left[- \frac{p \cdot y}{p \cdot e} + c\, \frac{\|Qp\|^2}{(p \cdot e)^2} \right] e + y - \frac{c}{p \cdot e}\, Qp$$

is a C^1 *individual excess demand function. Moreover,* $\zeta(y,c)(p) = \zeta(y,c)(q)$ *if and only if* q *is a multiple of* p.

Proof. This is entirely straightforward and left to the reader. We note in passing that $\zeta(y,c)$ actually arises from maximization of the quadratic utility function

$$u(x) = x \cdot e - \frac{1}{2}\, c \|Qx - y\|^2$$

and hence satisfies the strong axiom of revealed preference. ∎

If ζ: U → H satisfies the weak axiom (or, for that matter, the strong axiom) and the requirement that $\zeta(p) = \zeta(q)$ only if p is a multiple of q then, for any strictly positive function β: U → $(0,\infty)$, the function $\beta(p)\zeta(p)$ also satisfies the weak axiom (or the strong axiom if ζ does).

Lemma 4.5. *If* M, ϵ *are positive numbers and* ψ: $W \to [-M,+M]$ *is a continuous function defined on the open subset* W *of* H, *then there are strictly positive* C^1 *functions* Ψ_1, Ψ_2: $U \to [\epsilon/4, M+2\epsilon]$ *such that* $|\psi(x)-[\Psi_1(x)-\Psi_2(x)]| < \epsilon$ *for every* x *in* U.

Proof. Bonic and Frampton (1966) show that there is a C^1 function $\Psi:U \to \mathbb{R}$ such that $|\psi(x)-\Psi(x)| < \epsilon/4$ for each x in U. Define ξ: $\mathbb{R} \to \mathbb{R}$ by

$$\xi(t) = \frac{\epsilon}{4} \qquad \text{for } t < 0 ;$$

$$\xi(t) = \left(\frac{2}{\epsilon}\right) t^2 \qquad \text{for } 0 \leq t \leq \frac{\epsilon}{4} ;$$

$$\xi(t) = t + \frac{3\epsilon}{4} \qquad \text{for } t > \frac{\epsilon}{4} .$$

Set $\Psi_1(x) = \xi(\Psi(x))$, $\Psi_2(x) = \Psi_1(x)-\Psi(x)$; it is easily checked that Ψ_1, Ψ_2 have the desired properties. ∎

With these preliminaries out of the way, we can turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. We are going to decompose Z as an infinite sum of individual excess demand functions; it will then be easy to rewrite this sum as an integral. Fix a number ϵ with $0 < \epsilon < 1$. For p in U we have

$$Z(p) = \lambda(Z(p))e + \sum_{n=1}^{\infty} \sigma_n(QZ(p))e_n$$

where $\sigma(QZ(p)) = (\sigma_n(QZ(p))_{n=1}^{\infty}$. Since Z is bounded, Lemma 4.3 implies that $\sigma_n(QZ(p))$ is also bounded; so we may use Lemma 4.5 to find strictly positive C^1 functions σ_n, β_n: $U \to \mathbb{R}$ such that

$$|\sigma_n(QZ(p)) - [\sigma_n(p)-\beta_n(p)]| < 2^{-n}\epsilon$$

for each p in U. Write $K = \sup_{p \in U} \|Z(p)\|$. Recall that ς is the function defined in Lemma 4.4, and, for each n, set

$$c_n = 2^{-n}\epsilon,$$

$$\hat{u}_n(p) = \sigma_n(p)\varsigma(e_n, c_n)(p),$$

$$\hat{\beta}_n(p) = \beta_n(p)\varsigma(-e_n, c_n)(p).$$

Keep in mind that, for each p in U, $p \cdot e > \delta$ and that $\|p\| \leq M$ for some M. Straightforward estimates using the triangle inequality and the inequalities arising from our constructions show that

$$\|\hat{\alpha}_n(p)\| \leq |\sigma_n(QZ(p))| + 2^{-n}K(1+\rho)(1+2M\delta^{-1}+M^2\delta^{-2}), \tag{4.1}$$

and that $\|\hat{\beta}_n(p)\|$ admits the same bound. Since $\sigma_n(QZ(p))$ is the nth term of a sequence that is absolutely summable, we conclude that $\Sigma\|\hat{\alpha}_n(p)\| < \infty$ and $\Sigma\|\hat{\beta}_n\| < \infty$. Hence, we may define

$$Z^1(p) = \sum_{n=1}^{\infty} [\hat{u}_n(p) + \hat{\beta}_n(p)] \ ;$$

this series converges absolutely. By Lemma 4.4 and the remark following it, Z^1 is the infinite sum of individual excess demand functions.

We wish to estimate $\|Z(p)-Z^1(p)\|$. We first note that, after simplifying, we obtain

$$\begin{aligned}\hat{\alpha}_n(p) + \hat{\beta}_n(p) &= [\alpha_n(p) - \beta_n(p)]e_n \\ &\quad - [\alpha_n(p) - \beta_n(p)]\,\frac{p \cdot e_n}{p \cdot e} \\ &\quad - c_n[\alpha_n(p) + \beta_n(p)]\left[\frac{\|Qp\|^2}{(p \cdot e)^2} - \frac{1}{p \cdot e}\ Qp\right] \\ &= R_n^1(p) - R_n^2(p) - R_n^3(p)\ .\end{aligned}$$

The construction of α_n, β_n guarantees that

(4.2) $$\|R_n^1(p) - \alpha_n(QZ(p)\| < 2^{-n}\epsilon.$$

That $p \cdot Z(p) = 0$ (Walras's law), together with the above estimates, implies

(4.3) $$\|R_n^3(p)\| < 2^{-n}\epsilon[4K(1+\rho)(1+2M\delta^{-1}+M^2\delta^{-2})].$$

If we combine all three estimates, we obtain

$$\|Z(p) - Z^1(p)\| < \epsilon C$$

where C is a constant depending on M, ρ, and δ, but not on ϵ. Since ϵ is arbitrary, we can make $\|Z(p)-Z^1(p)\|$ as small as we wish.

Finally, note that, by writing $Z(p)-Z^1(p)$ as an infinite sum and applying the inequalities (4.2), (4.3), and (4.4), we can show that $Z(p)-Z^1(p)$ is the uniformly convergent sum of an infinite series of continuous functions, and is thus continuous.

To summarize, we have shown how to construct a sequence of individual excess demand functions whose sum Z^1 has the property that $\|Z-Z^1\| < \epsilon C$ and that $Z-Z^1$ is continuous. But then the function $Z-Z^1$ is continuous, is bounded, is homogeneous of degree zero, and satisfies Walras's law. Hence, we may simply repeat the above argument to obtain another sequence of individual excess demand functions with sum Z^2 such that $\|(Z-Z^1)-Z^2\| < 1/2\ \epsilon C$. Continuing in this way, we obtain a sequence of individual excess demand functions; the sum of the nth sequence is Z^n and

$$\|Z(p) - \sum_{j=1}^{n} Z^j(p)\| < (1/2)^{n+1}\ \epsilon C.$$

Hence, $Z = \Sigma_{j=1}^{\infty} Z^j$, and this series converges uniformly. After rewriting the double sequence of individual excess demand functions as a single sequence $\{z_i\}$, we see that $Z(p) = \Sigma_{j=1}^{\infty} z_i(p)$. Moreover, our construction shows that this series converges absolutely. (The crucial inequality is (4.2).)

It remains only to rewrite this sum as an integral. If we define

$$z_t(p) = 2^n z_n(p) \qquad \text{for} \quad 2^{-n} < t \leq 2^{-(n+1)},$$

then

$$\int z_t(p)dt = \sum_{n=1}^{\infty} z_n(p) = Z(p)$$

as required. ∎

5. Indeterminate Regular Equilibria

One of the implications of Theorem 4.1 is that economies with regular, indeterminate equilibria are very easy to produce if H is infinite-dimensional. In fact, we can easily exhibit economies with any prescribed degree of indeterminacy. The existence of such economies reflects the fact that on any infinite-dimensional Hilbert space H there are always continuous linear operators $S\colon H \to H$ that are onto but have kernels of any prescribed dimension. To construct such operators in our separable framework, fix an integer $k \geq 1$ and an orthogonal basis $\{e_j\}$ for H. Every x in H has a unique expression $x = \Sigma_{j=1}^{\infty} \alpha_j e_j$. We define $S_k\colon H \to H$ by

$$S_k\left(\sum_{j=1}^{\infty} \alpha_j e_j\right) = \sum_{j=k+1}^{\infty} \alpha_{j-k} e_j .$$

If H is the space ℓ_2 of square-summable sequences and $\{e_j\}$ is the usual orthonormal basis, the operator S_k just shifts every sequence backward by k steps; that is,

$$S_k(\alpha_1,\alpha_2,\ldots) = (\alpha_{k+1},\alpha_{k+2},\ldots).$$

It is evident that the operator S_k is onto and has a k-dimensional kernel.

To obtain indeterminate, regular equilibria, all we need to do is to turn such linear operators into excess demand functions.

<u>Theorem 5.1</u>. *Assume that the commodity space* H *is infinite-dimensional and let* k *be a positive integer. Then there are non-empty open sets* $U' \subset U$ *and* $W \subset \mathcal{E}$ *such that, for each economy* z *in* W, *the set of equilibrium prices of* z *in* U' *is a non-empty k-dimensional manifold and consists entirely of regular equilibria.*

<u>Proof</u>. Let e be the distinguished unit vector in the price domain U, and write E_1 for the orthogonal complement of e. We can write each vector x in H uniquely as $x = x_0+x_1$, where $x_0 = (x\cdot e)e$ and $x_1 = x-(x\cdot e)e$ belongs to E_1. Let $A\colon E_1 \to E_1$ be any continuous linear operator which is onto and has a (k-1)-dimensional kernel. We now define a function $Z\colon U \to H$ by

$$Z(p) = \frac{1}{p\cdot e} Ap_1 - \frac{1}{(p\cdot e)^2}(p_1\cdot Ap_1)e.$$

Since $p\cdot e > \delta$ for each p in U, Z is well defined; it is easily seen to be bounded, C^1, homogeneous of degree zero, and to satisfy Walras's law. By Theorem 4.1, there is an economy z that has Z as its mean excess demand function. Since $Z(e) = 0$ and $DZ(e)q = Aq_1$ for each q in H, e is a regular equilibrium price of the economy z. Since kernel A is a $(k-1)$-dimensional subspace of E_1 and $DZ(e)e = 0$, it follows that kernel $DZ(e)$ is a k-dimensional subspace of H. Proposition 3.1 now provides the required open sets U', W. ∎

6. C^1-Integrable Economies and Determinate Equilibria

In this section, we shall see that the picture obtained in Section 5 changes dramatically if we impose an additional requirements, that the derivatives of individual excess demand should have finite mean. We call this C^1 integrability. Indeed, the behavior of C^1-integrable economies is qualitatively very much like that of finite-dimensional economies. Of course, this means that C^1 integrability is not a minor technical condition. Rather, it is the mathematical expression of a simple underlying economic requirement, that the family of individual demand functions represented in the economy not be too diverse.

6.A. C^1 Integrability

By Lemma A.1 in the Appendix, it follows that the function $t \to \sup_{p\in U} \|Dz_t(p)\|: \ [0,1] \to [0,\infty]$ is measurable. We now consider strengthening this condition:

Definition 6.1. The economy z is *C^1-integrable* if

$$\int \sup_{p\in U} \|Dz_t(p)\| dt.$$

We denote the set of C^1-integrable economies by $\mathcal{E}^*$.

Observe that the economy z is certainly C^1-integrable if there is a constant M such that $\|Dz_t(p)\| \leq M$ for each p in U and t in $[0,1]$.

Proposition 6.1.

(i) *If the economy* z *is* C^1*-integrable then*

$$\int \sup_{p\in U} \|z_t(p)\| dt < \infty.$$

(ii) *If the economy* z *is* C^1*-integrable, then it has smooth demand. Moreover, if* $p \in U$ *and* $q \in H$, *then*

$$DZ(p)q = \int Dz_t(p)q\ dt.$$

(iii) *The set* $\mathcal{E}^*$ *of* C^1*-integrable economies is a non-empty open and closed subset of* $\mathcal{E}$.

Notice that, in cases where H is finite-dimensional or where the population falls into a finite number of types, the C^1 integrability condition amounts to a minor technical condition of a standard variety.

Proof.

(i) Fix a price p^* in U and t in $[0,1]$. For any p in U, write $p = p^*+(p-p^*)$. Apply the mean value theorem (Lang (1982)) and the triangle inequality to conclude that

$$\|z_t(p)\| \leq \|z_t(p^*)\| + \|p-p^*\| \sup_{\bar{p}\in U} \|Dz_t(\bar{p})\|.$$

Since U is bounded, there is a constant C such that $\|p-p^*\| \leq C$ for all p in U. We then obtain

$$\int \sup_{p\in U} \|z_t(p)\|dt \leq \int\|z_t(p^*)\|dt + C \int \sup \|Dz_t(\bar{p})\|dt.$$

The first of these integrals is finite (because the mapping $T \to z_t(p^*)$ is Bochner integrable) and the second is finite by assumption. This is (i).

(ii) This follows by exactly the same argument as Proposition 2.3(i).

(iii) That $\mathcal{E}^*$ is open and closed follows exactly as in Proposition 2.3. Since the zero economy is C^1-integrable, $\mathcal{E}^*$ is not empty. ∎

6.B. Semi-Fredholm Operators

In the next subsection, we shall spell out the important consequences of Proposition 6.1 for the determinacy of regular equilibria. In this subsection, we collect some mathematical notions and facts we shall need.

Definition 6.2. The continuous linear operator $T: H \to H$ has finite rank if range T is a finite-dimensional subspace of H. The operator T is compact if the closure of the image of each bounded set in H is compact.

Fact 6.1. The continuous linear operator T is compact if and only if it is the norm limit of finite rank operators (Dunford and Schwartz (1958)).

Fact 6.2. For each t in [0,1], let $A_t: H \to H$ be a continuous linear operator. Assume that

(i) for each x in H, the map $t \to A_t x$ is integrable;

(ii) $\int \|A_t\| dt < \infty$. (Notice that measurability of $t \to \|A_t\|$ follows as in Lemma A.1.) Then the mapping $A: H \to H$ given by

$$Ax = \int A_t x \, dt$$

is a continuous linear operator, and $\|A\| \leq \int \|A_t\| dt$. Moreover, if each A_t is a compact operator, so is A (Berger and Coburn (1985, Lemma 12)).

Definition 6.3. Let $T: H \to H$ be a continuous linear operator. The corange of T is the subspace

$$\text{corange } T = \{h: h \in H \text{ and } h \cdot Tx = 0 \text{ whenever } x \in H\}.$$

The operator T is semi-Fredholm if range T is closed and at least one of the subspaces, kernel T, corange T is finite-dimensional. In that case, we define the index of T by

$$\text{index } T = \dim(\text{kernel } T) - \dim(\text{corange } T).$$

The index is an integer, $+\infty$ or $-\infty$. We denote the set of semi-Fredholm operators of index s by $\mathcal{F}_s(H)$. The space of all bounded linear operators on H is denoted by $\mathcal{L}(H)$.

If the operator T is semi-Fredholm and both the kernel and the corange of T are finite-dimensional, the operator T is called Fredholm. Equivalently, an operator T is Fredholm when it is semi-Fredholm of finite index. Note that, if H is finite-dimensional, then every operator on H is Fredholm of index zero.

Fact 6.3. The set $\mathcal{F}_s(H)$ of semi-Fredholm operators of index s is open in the space $\mathcal{L}(H)$ (Atkinson (1951) and Yood (1985)).

Fact 6.4. If A is a semi-Fredholm operator of index s and K is a compact operator, then A+K is a semi-Fredholm operator of index s (Atkinson (1951) and Yood (1951)).

Fact 6.5. If A is negative quasi-semi-definite (that is, $x \cdot Ax \leq 0$ for every x) and semi-Fredholm, then index $A = 0$.

Proof. For each $\epsilon > 0$, set $A_\epsilon = A - \epsilon I$. then $A_\epsilon x = Ax - \epsilon x$ for each x, and $x \cdot A_\epsilon \leq -\epsilon \|x\|^2$ for each x. This implies, in particular, that $\|A_\epsilon x\| \geq \epsilon \|x\|$ for each x; hence A_ϵ is an invertible operator, and thus is semi-Fredholm of index zero for each ϵ. Since $A = \lim_{\epsilon \to 0} A_\epsilon$ and A is semi-Fredholm, Fact 6.3 implies that index $A = 0$. ∎

6.C. Determinacy of Regular Equilibria

Proposition 6.1 shows that, for a C^1-integrable economy, the derivative of mean excess demand is the integral of the derivatives of individual demand. This enables us to use the facts about compact and semi-Fredholm operators, together with the basic implication of the weak axiom (Proposition 2.1), to prove that regular equilibria of C^1-integrable economies are determinate.

The idea of the proof is as follows: Because of the weak axiom the derivative of individual excess demand function is a finite rank perturbation of a negative quasi-semi-definite operator. The aggregate of negative quasi-semi-definite operators is negative quasi-semi-definite; the aggregate of finite rank operators is compact. Consequently, at a regular equilibrium the derivative of mean excess demand is semi-Fredholm of index zero. Since regularity means that the corange is one-dimensional, the kernel must also be one-dimensional. Hence the equilibrium is determinate.

Theorem 6.2. *Let* z *be a* C^1*-integrable economy and* p *a regular equilibrium price for* z. *Then* p *is determinate.*

Proof. Regularity of the equilibrium price p means that the range of DZ(P) coincides with $T_p = \{h: h \in H, h \cdot p = 0\}$, which is a closed subspace of H. Moreover, the corange of T is the subspace spanned by p, which is one-dimensional. Hence DZ(p) is a semi-Fredholm operator and

$$\begin{aligned} \text{index } DZ(p) &= \dim(\text{kernel } DZ(p)) - \dim(\text{corange } DZ(p)) \\ &= \dim(\text{kernel } DZ(p)) - 1. \end{aligned}$$

Since p is determinate exactly when dim(kernel DZ(p)) = 1, the theorem follows if we can show that index DZ(p) = 0. To do this, we show that DZ(p) is the sum of two negative quasi-semi-definite operators and a compact operator.

Proposition 6.1, implies that, for each h in H,

$$DZ(p)h = \int Dz_t(p)h \, dt.$$

Let $J = \{t: t\in[0,1], z_t(p)\neq 0\}$, $J' = [0,1]\backslash J$. Since $t \to z_t(p)$ is measurable, both J and J' are measurable sets. We define continuous linear operators $A, A': H \to H$ by

$$Ah = \int_J Dz_t(p)h\, dt,$$

$$A'h = \int_{J'} Dz_t(p)h\, dt.$$

For each t in J', $z_t(p) = 0$, so $h \cdot z_t(p) = 0$ for every h in H. Hence, by Proposition 2.1, $h \cdot Dz_t(p)h \leq 0$ for T in J' and h in H, whence

$$h\cdot A'h = \int_{J'} h\cdot Dz_t(p)h\, dt \leq 0.$$

In other words, A' is negative quasi-semi-definite.

Now, for each t in J, $z_p(t) \neq 0$. If $h \in H$, set

$$P_t h = \frac{h\cdot z_t(p)}{\|z_t(p)\|^2}\ z_t(p),$$

$$Q_t h = h - P_t h,$$

so that P_t is the orthogonal projection onto the space spanned by $z_t(p)$, and $Q_t h$ is the orthogonal projection onto the orthogonal complement of this space; note that $P_t + Q_t = I$, the identity operator. For each h in H, we have:

$$h \cdot [Q_t(Dz_t(p))Q_t]h = (Q_t h) \cdot Dz_t(p)(Q_t h) \leq 0$$

since Q_t is symmetric and $Q_t h$ is orthogonal to $z_t(p)$. In other words, the operator $Q_t(Dz_t(p))Q_t$ is negative quasi-semi-definite for each t in J. Since $P_t + Q_t = I$, we have

$$Dz_t(p) = Q_t(Dz_t(p))Q_t + P_t(Dz_t(p)) + Q_t(Dz_t(p))P_t.$$

Since P_t is a rank one projection, the second and third operators on the right-hand side have rank at most one.

Observe that

$$\|Q_t(Dz_t(p))Q_t\| \leq \|Dz_t(p)\| \ \|Q_t\|^2 \leq \|Dz_t(p)\|$$

since Q_t is an orthogonal projection. Hence, we may define operators $B,K\colon\ H \to H$ by

$$Bh = \int_J Q_t(Dz_t(p)Q_t h \ dt,$$

$$Kh = \int_J [P_t(Dz_t(p))h + Q_t(Dz_t(p))P_t]h \ dt.$$

Fact 6.2 implies that B is a continuous linear operator and K is a compact, continuous linear operator. Because $Q_t(Dz_t(p))Q_t$ is negative quasi-semi-definite (for each t in J), we see as before that B is also negative quasi-semi-definite.

We have thus obtained the decomposition

$$DZ(p) = A' + B + K,$$

where A' and B are negative quasi-semi-definite and K is compact. Since $A'+B = DZ(p)-K$, and $DZ(p)$ is semi-Fredholm, so is $A'+B$. Since $A'+B$ is negative quasi-semi-definite semi-Fredholm, Fact 6.5 implies that $\text{index}(A'+B) = 0$, so, by Fact 6.4, $\text{index } DZ(p) = \text{index}(A'+B) = 0$, which is the desired result. ∎

6.D. <u>Density of Regular Economies</u>

We conclude by showing that, within the class of C^1-integrable economies, the regular economies are dense.

Definition 6.4. Let V be an open, connected subset of H and $f\colon V \to H$ a C^1 function. The vector h in H is a regular value of f if $Df(x)$ is onto for every x with $f(x) = h$.

Fact 6.6. Let V be an open, connected subset of H and let $f\colon V \to H$ be a C^1 function with the property that $Df(x)$ is Fredholm of index zero for each x in V. Then the set of regular values of f is dense in H. (See Smale (1965); this is a consequence of Sard's theorem.)

Theorem 6.3. *The set of regular C^1-integrable economies is dense in the set of all C^1-integrable economies.*

Before giving the proof of the theorem, we establish an important technical lemma. It allows us, without loss of generality, to assume that the derivative of mean excess demand is Fredholm of index zero.

Lemma 6.4. *If z is a C^1-integrable economy and $\epsilon > 0$ then there is a C^1-integrable economy z^* such that $d(z,z^*) < \epsilon$ and $DZ^*(p)$ is Fredholm of index 0 for all p in U.*

Proof. The idea of the proof is that $DZ(p)$ is the sum of a negative quasi-semi-definite and a compact operator as we showed in the proof of Theorem 6.2. By changing the demand of a small subset of consumers, we can ensure that the negative quasi-semi-definite operator is actually negative quasi-definite, and consequently, non-singular. This implies that it is semi-Fredholm with index zero, and consequently, so is $DZ(p)$ by Fact 6.4.

By our standing assumption, there is a vector e in U such that $\|e\| = 1$ and a positive number δ such that $p \cdot e > \delta$ for each p in U. Again let E_0 be the one-dimensional space spanned by e; let E_1 be its orthogonal complement; and let Q be the orthogonal projection onto E_1. Define the C^1 function ζ: $U \to H$ by

$$\zeta(p) = \frac{\|Qp\|^2}{2\lambda(p)^2} e - \frac{1}{2\lambda(p)} Qp.$$

It is easy to see that ζ is an individual excess demand function. In fact, ζ is the excess demand function arising from the utility function $u(x) = \lambda(x) - \|Qx\|^2$ and initial endowment e. This construction is the same as that in Lemma 4.4, with $y = 0$.

Since U is a bounded set and $\lambda(p) = p \cdot e > \delta$ for each p in U, $\sup_{p \in U} \|\zeta(p)\|$ is finite. We claim that the derivative of ζ has the following properties:

(i) $\sup_{p \in U} \|D\zeta(p)\| < \infty$;

(ii) $QD\zeta(p)Q = -\frac{1}{\lambda(p)} Q$, for each p in U.

The easiest way to see that these properties hold is by direct computation. Choose an orthonormal basis $e_1, e_2, \ldots$ for E_1; then every x in H may be written uniquely as

$$x = \lambda(x)e + \sum_{i=1}^{\infty} \lambda_i(x)e_i$$

where $\lambda_1(x)$, $\lambda_2(x), \ldots$ are real numbers. Then

$$\|Qp\|^2 = \sum_{i=1}^{\infty} \lambda_i(p)^2,$$

so that

$$\varsigma(p) = \frac{\sum_{i=1}^{\infty} (p)^2}{2\lambda(p)^2} \; p - \frac{1}{2\lambda(p)} \sum_{i=1}^{\infty} \lambda_i(p) e_i .$$

Hence, if we write the matrix for $D\varsigma(p)$ relative to the orthonormal basis $e, e_1, e_2, \ldots$ for H, we obtain

$$D\varsigma(p) = \begin{bmatrix} \frac{-\Sigma\lambda_i(p)^2}{\lambda(p)^3} & \frac{\lambda_1(p)}{2\lambda(p)^2} & \frac{\lambda_2(p)}{2\lambda(p)^2} & \cdots \\ \frac{\lambda_1(p)}{\lambda(p)^2} & -\frac{1}{2\lambda(p)} & 0 & \cdots \\ \frac{\lambda_2(p)}{\lambda(p)^2} & 0 & -\frac{1}{2\lambda(p)} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix},$$

which yields (i) and (ii).

We now observe that, by assumption $\int_{p \in U} \|Dz_t(p)\| dt < \infty$. Consequently, by Proposition 6.1(i), we can find a positive number α so small that $0 < \alpha < 1/2$ and

$$\int_{[0,\alpha]} [\sup_{p\in U} \|z_t(p)\| + \sup_{p \in U} \|Dz_t(p)\|] dt < \epsilon/2.$$

We define the economy by

$$z_t^*(p) = c\varsigma(p) \qquad \text{for } 0 \le t \le \alpha$$

$$z_t^*(p) = z_t(p) \qquad \text{for } \alpha < t \le 1$$

where c is a very small positive constant. It is evident that z^* is a C^1-integrable economy and that $d(z,z^*) < \epsilon$ if c is sufficient small.

We claim that $DZ^*(p)$ is a Fredholm operator of index zero for each p in U. To see this, set

$$W(p) = \int_{[\alpha,1]} z_t(p)dt$$

and note that $Z^*(p) = \alpha c\zeta(p)+W(p)$, so that $DZ^*(p) = \alpha cD\zeta(p)+DW(p)$. Hence,

$$Q(DZ^*(p))Q = \alpha cQ(D\zeta(p))Q + Q(DW(p))Q$$

$$= - \frac{ac}{2\lambda(p)} Q + Q(DW(p))Q.$$

As we have shown in the proof of Theorem 6.2, $DW(p)$ is the sum of a negative quasi-semi-definite operator and a compact operator, so $Q(DW(p))Q$ is also the sum of a negative quasi-semi-definite operator and a compact operator. Thus, if we regard $Q(DZ^*(p))Q$ as an operator from E_1 = range Q to itself, we conclude that it is the sum of a negative multiple of the identity operator, a negative quasi-semi-definite operator, and a compact operator. Hence, $Q(DZ^*(p))Q$ is the sum of a negative quasi-semi-definite operator and a compact operator. Since negative quasi-definite operators are invertible, and in particular are Fredholm of index zero, we conclude from Fact 6.4 that $Q(DZ^*(p))Q$ is also Fredholm of index zero. Since $DZ^*(p)$ and $Q(DZ^*(p))Q$ differ by an operator whose rank is at most two, we again conclude from Fact 6.4 that $DZ^*(p)$ is Fredholm of index zero, for each p. ∎

<u>Proof of Theorem 6.3</u>. By Lemma 6.4, we may replace our given economy by an economy z^* for which $DZ^*(p)$ is Fredholm of index zero for each p. Moreover, as in the proof of Lemma 6.4, we may as well assume that (for some α, with $0 < \alpha < 1/2$) $z_t^*(p) = 0$ for all p and $0 \le t \le \alpha$. Choose $0 < \alpha' \le \alpha$.

To approximate z^* by a regular economy, we consider the restriction of QZ^* to E_1. Regarded as a map of E_1 to itself, QZ^* is certainly C^1, and its derivative at p is $D(QZ^*)(p) = Q(DZ^*(p)) = Q(DZ^*(p))Q$ (regarded as a map from E_1 to itself), which is Fredholm of index zero. By Fact 6.6, QZ^* has regular values h in E_1 arbitrarily close to 0. We now define the economy $\hat{z}$ by

$$\hat{z}_t(p) = \frac{1}{\alpha}(-h + \frac{p\cdot h}{p\cdot e}e) \qquad 0 \leq t \leq \alpha'$$

$$\hat{z}_t(p) = z^*_t(p) \qquad \text{if} \quad \alpha' < t \leq 1.$$

Certainly, $\hat{z}$ is a C^1-integrable economy. Moreover, $d(z^*,\hat{z}) < \epsilon/2$ if α' and $\|h\|$ are sufficiently small, so $d(z,\hat{z}) < \epsilon$ if α, α' and $\|h\|$ are sufficiently small. To see that $\hat{z}$ is a regular economy, suppose that p is an equilibrium price, so that $\hat{Z}(p) = 0$. As we have noted, Walras's law implies that the range of $D\hat{Z}(p)$ is a subspace of T_p; to show that p is a regular equilibrium price we need to show that range $D\hat{Z}(p) = T_p$. If this were not so, then range $D\hat{Z}(p)$ would be a proper subspace of T_p; since $p\cdot e \neq 0$, this would imply that $Q(\text{range } D\hat{Z}(p))$ would be a proper subspace of $Q(T_p)$, and hence of $E_1 = \text{range } Q$. On the other hand, for each q in U,

$$\hat{Z}(p) = \int \hat{z}_t(q)dt$$

$$= \int_{[\alpha',1]} z^*_t(q)dt + \int_{[0,\alpha']} \hat{z}_t(q)dt$$

$$= Z^*(q) - h + \frac{q\cdot h}{q\cdot e}e.$$

Hence,

$$Q\hat{Z}(q) = QZ^*(q) - Qh + \frac{q\cdot h}{q\cdot e} = QZ^*(q) - h.$$

If we differentiate, we obtain

$$Q(D\hat{Z}(p)) = D(Q\hat{Z}(p)) = D(QZ^*(p)).$$

Moreover, since $\hat{Z}(p) = 0$, $QZ^*(p) = h$. Since h is a regular value of QZ^*, $Q(D\hat{Z}(p)) = D(QZ^*(p))$ maps $E_1 = \text{range } Q$ onto E_1. Hence, $Q(\text{range}D\hat{Z}(p))$ is not a proper subspace of E_1, so $\text{range}D\hat{Z}(p)$ is not a proper subspace of T_p; that is,

$$\text{range}D\hat{Z}(p) = T_p.$$

Consequently, p is a regular equilibrium price. ∎

We recall that, in the finite-dimensional framework, the set of regular economies is open. Because of the non-compactness of the price space it is no longer the case in the infinite-dimensional setting. We do not know the true topological nature of the set of regular economies.

7. Economies Without Smooth Demand

If the commodity space is finite dimensional, or there are finitely many types, or, more generally, the economy is C^1-integrable, then aggregate excess demand is smooth, and, for a dense subset of such economies, equilibria are determinate. Without C^1 integrability, there are three additional possibilities: aggregate excess demand may be smooth and have regular indeterminate equilibria, it may fail to have regular equilibria, or it may not be smooth. About the case of smooth demand without regular equilibria we know nothing. Non-smooth demand economies may, however, exhibit a robust indeterminacy of a rather different type than that discussed in Section 5.

As a preliminary, we need the fact that there are continuous maps with the property that if they intersect a small C^1 function once, they intersect it infinitely many times.

<u>Lemma 7.1</u>. *If* $U \subset H$ *is open, then there is a continuous function* $f\colon U \to H$, *and an* $\epsilon > 0$, *such that, if* $g\colon U \to H$ *is any* C^1 *function with* $\sup_p \|g(p)\|$, $\sup_p \|Dg(p)\| < \epsilon$ *and* $f(p) = g(p)$ *for some* p, *then this is true for infinitely many values of* $p \in U$.

<u>Proof</u>. Write $H = E_0 + E_1$ where E_0 is one-dimensional, $e \in E_0$, and E_1 is orthogonal to E_0. Let $p_0 e$ and p_1 be the corresponding components of p. For any function $f^*\colon \mathbb{R} \to \mathbb{R}$ define

$$f(p) = f^*(p_0)e + p_1.$$

Suppose that f^* has the property described in the lemma for C^1 functions $g^*\colon \mathbb{R} \to \mathbb{R}$ with $\sup \|g^*\|$, $\sup \|Dg^*\| < \delta$. Such functions exist since, for example, the sample paths of a Brownian motion are almost surely of this type. We claim that f also has the required property.

To see this, choose g as in the statement of the Lemma, and consider solutions to the equation $f(p) = g(p)$, or equivalently

$$f^*(p_0) = g_0(p_0,p_1),\ p_1 = g_1(p_0,p_1).$$

Observe that, since $\|Dg\| < 1$, $\|Dg_1\| < 1$, and by the implicit function theorem $g_1(p_0,p_1)-p_1 = 0$ has a unique C^1 solution $p_1(p_0)$. Hence, $f(p) = g(p)$ if and only if $f^*(p_0) = g_0(p_0,p_1(p_0))$. By assumption on g, $g_0(p_0,p_1(p_0))$ is the composition of C^1 functions. Moreover, $\sup \|g_0\| \le \sup \|g\| < \epsilon$, and by the implicit function theorem

$$\sup \|Dg_o\| \le \sup \|D_0 g_0 - D_1 g_0 (D_1 g_1 - I)^{-1} D_0 g_1\| \le 3\epsilon/(1-\epsilon).$$

Consequently, if ϵ is chosen so that $3\epsilon/(1-\epsilon) = \delta$, the infinite dimensional version of the lemma follows from the one dimensional version. ∎

Clearly we can use Walras's law, homogeneity, and the results of Section 4 to construct an economy with excess demand $Z: U \to H$ where $U = U_0 + U_1$, where $U_0 \subset E_0$, $U_1 \subset E_1$ are open, $e \in E_0$, and E_1 is orthogonal to E_0 with property that $Z_1: U_1 \to E_1$ is as in the lemma. Moreover, $Z(p) = 0$ if and only if $Z_1(p) = 0$. Suppose $\|Z-Z'\| < \epsilon$, where ϵ is as in the lemma. Set $g = Z_1 - Z_1'$. Then $Z'(p) = 0$ if and only if $Z_1(p) = g(p)$. In other words, since g is C^1, $Z'(p) = 0$ infinitely many times in U or not at all. With a small amount of additional work, we can arrange Z on the boundary of U so that Z and Z' must have equilibria, and consequently we have a robust example of a different kind of indeterminacy.

We must emphasize that this example is also a valid example of a robust indeterminacy in a finite economy. We can equally well construct a continuous, but not C^1, excess demand function based on underlying preferences of consumers, for which no small C^1 perturbation can eliminate the infinitely many equilibria. The only difference is that, with finite economies, smoothness of individual preferences is enough to rule out such an example. In a large square economy, the assumption that aggregate excess demand is C^1 is an assumption not only on individual demand, but also on the relationship between the characteristics of different individuals. More strongly, Theorem 4.1 shows that, with a continuum of consumers, every continuous excess demand function arises from underlying smooth consumers.

APPENDIX

Recall that for $F\colon U \to H$, a C^1 mapping, the (Frechet) derivative of F at p is a continuous linear operator $DF(p)\colon H \to H$; we write $DF(p)q$ for the value of $DF(p)$ at q. Then the norm of $DF(p)$ is its norm as a linear operator, that is,

$$\|DF(p)\| = \sup\{\|DF(p)q\|\colon q \in H, \|q\| \le 1\}.$$

Lemma A.1. *Let* $f\colon U\times[0,1] \to H$ *be a mapping such that*

(i) $f_t(\cdot) = f(\cdot,t)\colon U \to H$ *is* C^1 *for each t;*

(ii) $f(p,\cdot)\colon [0,1] \to H$ *is measurable for each* p.

Then the following mappings are all measurable (for each p_0 *in* U *and* q_0 *in* H):

(a) $t \to \|f_t(p_0)\|\colon [0,1] \to H;$

(b) $t \to \sup_{p\in U} \|f_t(p)\|\colon [0,1] \to [0,\infty];$

(c) $t \to Df_t(p_0)q_0\colon [0,1] \to H;$

(d) $t \to \|Df_t(p_0)\|\colon [0,1] \to [0,\infty]$

(e) $t \to \sup_{p\in U} \|Df_t(p)\|\colon [0,1] \to [0,\infty].$

Proof. Let $B = \{q\colon q\in H \text{ and } \|q\| \le 1\}$ be the unit ball of H. Since H is separable, we can choose a countable dense subset $\{q_n^*\}$ of B, and a countable dense subset $\{p_n^*\}$ of H. Now,

$$\|f_t(p_0)\| = \sup_{q\in B} q\cdot f_t(p_0) = \sup q_n^*\cdot f_t(p_0).$$

By assumption, $t \to f_t(p_0)$ is measurable; since the inner product is continuous, $t \to q_n^*\cdot f_t(p_0)$ is measurable for each n. Hence, the function $t \to \|f_t(p_0)\|$ is the supremum of a countable family of measurable mappings, and is measurable. This is (a).

To prove (b), we simply observe that, because f_t is continuous for each t,

$$\sup_{p \in U} \|f_t(p)\| = \sup_n \|f_t(p_n^*)\|.$$

Hence, the function $t \to \sup_{p \in U} \|f_t(p))\|$ is the supremum of the countable family of measurable functions $t \to \|f_t(p_n^*)\|$, and is thus measurable.

To prove (c), we write that, for fixed p_0, q_0, t,

$$Df_t(p_0)q_0 = \lim_{\lambda \to 0} \frac{f_t(p_0+\lambda q_0)-f_t(p_0)}{\lambda}$$

$$= \lim_{n \to \infty} \frac{f_t(p_0+\frac{1}{n}q_0) - f_t(p_0)}{(1/n)}$$

Clearly, then, the mapping $t \to Df_t(p_0)q_0$ is the limit of a sequence of measurable mappings, and is thus measurable.

Parts (d) and (e) are proved exactly as (a) and (b). ∎

REFERENCES

Abraham, R. and J. Robbin (1967), _Transversal Mappings and Flows_, Benjamin, N.Y.

Araujo, A. (1984), "Smooth Demand Functions in Infinite Dimensional Spaces," Instituto de Matematica Pura e Aplicada.

Atkinson, F.V. (1951), "The Normal Solubility of Linear Equations in Normed Spaces," _Mat. Sbornik_, (New Series) 28, 3-14.

Bartle, R.B. and L.M. Graves (1952), "Mappings Between Function Spaces," _Transactions of the American Mathematical Society_, 72, 400-413.

Berger, C.A. and L.A. Coburn (1985), "Toeplitz Operators on the Segal-Bargmann Space," Mathematical Sciences Research Institute Paper 01311-85.

Bonic, R. and J. Frampton (1966), "Smooth Functions on Banach Manifolds," _Journal of Mathematics and Mechanics_, 15, 877-898.

Debreu, G. (1970), "Economies with a Finite Set of Equilibria," _Econometrica_, 38, 387-392.

Diestel, J. and J. Uhl (1977), _Vector Measures_, Mathematical Surveys No. 15, A.M.S., Providence, RI.

Dunford, N. and J.T. Schwartz (1958), _Linear Operators_, Part I, Interscience, New York.

Kehoe, T.J. (1988), _Regularity and Index Theory for Economic Equilibrium Models_, forthcoming.

Kehoe, T.J. and D.K. Levine (1985), "Comparative Statics and Perfect Foresight in Infinite Horizon Economics," _Econometrica_, 53, 433-453.

Kehoe, T.J., D.K. Levine, and P.M. Romer (1988), "Determinacy of Equilibria in Dynamic Models With Finitely Many Consumers," unpublished manuscript.

Lang, S. (1962), Introduction to Differential Manifolds, Interscience, New York.

Mas-Colell, A. (1985), The Theory of General Economic Equilibrium: A Differentiable Approach, Cambridge University Press, Cambridge.

Michael, E. (1956), "Continuous Selections I.," Annals of Mathematics, 63, 361-382.

Muller, W.J. and M. Woodford (1986), "Stationary Overlapping Generations Economies with Production and Infinite-Lived Consumers," Journal of Economic Theory, forthcoming.

Ostroy, J. (1984), "The Existence of Walrasian Equilibrium in Large-Square Economies," Journal of Mathematical Economics, 13, 143-163.

Samuelson, P.A. (1958), "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money," Journal of Political Economy, 66, 467-82.

Shafer, W. and H. Sonnenschein (1981), "Market Demand and Excess Demand Functions," Handbook of Mathematical Economics, vol. II, ed. by K.J. Arrow and M.D. Intriligator, North-Holland, Amsterdam, 672-693.

Smale, S. (1965), "An Infinite-Dimensional Version of Sard's Theorem," American Journal of Mathematics, 87, 861-866.

Yood, B. (1951), "Properties of Linear Transformations Preserved Under Addition of a Completely Continuous Transformation," Duke Mathematical Journal, 599-612.