The Reputation Trap[1]

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Abstract

Few want to do business with a partner who has a bad reputation. Consequently once a bad reputation is established it can be difficult to get rid of. This leads on the one hand to the intuitive idea that a good reputation is easy to lose and hard to gain. On the other hand it can lead to a strong form of history dependence in which a single beneficial or adverse event can cast a shadow over a very long period of time. It gives rise to a reputational trap where an agent rationally chooses not to invest in a good reputation because the chances others will find out is too low. Never-the-less the same agent with a good reputation will make every effort to maintain it. Here a simple reputational model is constructed and the conditions for there to be a unique equilibrium that constitutes a reputation trap are characterized.

[1] I am indebted to Drew Fudenberg, Juan Block, Rohan Dutta, Mustafa Kaba, John Mair, Andrea Mattozzi, Salvatore Modica, Guillermo Ordonez, to the 24th Spring Meeting of Young Economists, and to the Lancaster Game Theory Conference. Financial support from the EUI Research Council is gratefully acknowledged.

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"Glass, china, and reputation are easily cracked, and never well mended."

sometimes attributed to Benjamin Franklin.

1. Introduction

It is conventional to think that a good reputation is easy to lose and hard to gain. One reason we suspect this might be the case is that if you have a good reputation people will be eager to do business with you – hence if they are cheated it will quickly become known. On the other hand if you have a bad reputation few will do business with you so even if you are honest few will find out. In such a setting it is intuitive that history matters. If an adverse event causes a loss of reputation the difficulty of restoring it provides little incentive for honesty, so the bad reputation will deservedly remain so long after the circumstances that caused it are gone. On the other hand, there are reasons for honesty besides reputation – if circumstances dictate honesty it will take a long time before others find out, but once they do reputation will be restored – and even after the circumstances dictating honesty are gone it will be desirable to continue to be honest to avoid losing reputation. In other words, once reputation is restored it will also persist. Consequently, two otherwise identical individuals may find themselves with entirely different incentives for honesty because of an adverse or beneficial event that happened in the distant past. There is, as we shall see, a rather important hole in this intuition.

The goal of this paper is to develop a model that captures the intuitive idea that differential observability leads to history dependence. In doing so we draw key elements from the reputational literature. Following the gang-of-four[3] we augment the "normal" type with behavioral types – as in Mailath and Samuelson [2001] these types are persistent but not completely so. We allow for good types (beneficial events) as in the gang-of-four and bad types (adverse events) as in Mailath and Samuelson [2001] and Horner [2002].[4] Finally, following an idea in Fudenberg and Levine [1989] we assume that the short-run players face an entry decision and that the information generated about long-run player behavior is greater if the short-run player chooses to enter than if not. This observational asymmetry leads to an important change from the Mailath and Samuelson [2001] model where good and bad events are symmetric and reputation is equally easily lost or restored.

This model captures the intuitive elements of persistent reputation if we add an additional assumption concerning the short-run player. As is standard in these types of models in each period a single representative short-run player is a stand-in for a large population of players. This implies myopic behavior. It also makes it difficult to coordinate a response to events in the distant past. As

[3]Kreps and Wilson [1982] and Milgrom and Roberts [1982]

[4]As Cripps, Mailath and Samuelson [2004] show this is essential if we are to have reputations restored as well as decline.

indicated without an assumption in this direction, there is a hole in the basic intuition of the first paragraph. If short-run players stay out and no information is generated it eventually becomes likely that the long-run player has migrated back to a "normal" type. It is now possible for the short-run players and long-run player to coordinate. On a particular date it is common knowledge that if the long-run player is normal honest behavior will take place and that the short-run player will enter. This is then a self-fulfilling prophecy.[5] It is not, however, a very compelling one: it requires that both players agree about the exact timing of events in the long-distant past and that they agree that "today is the day." To rule this out we assume that agents know only about events that took place during their lifetime and that short-run player strategies and beliefs are independent of calendar time.

Reputation theory has wide application to a variety of settings, but an important motivation for this research is a puzzle in the political economy of culture and institutions concerning the persistence of dysfunctional cultures. On the one hand there is a substantial literature indicating that these can be quite persistent. Acemoglu and Robinson [2001] give evidence for persistence on the order of four centuries. Bigoni et al [2013] have evidence of a similar effect over nearly nine centuries. Dell and Querubin [2018] have highly persuasive evidence for persistence on the order of a century and a half.

On the other hand it cannot be that it is simply impractical to change social and cultural norms: side by side with the survival of dysfunctional norms we see abrupt change over periods of a few decades. Two central aspects of culture are religion and language. Yet we observe that even these fundamental aspects of society change over short periods of time. Prior to 1990 the country of Ireland could well be described as Catholic. By the end of the decade the church lost its central place in Irish life and the country could be better described as secular.[6] With respect to language we may point to the remarkable example of Hebrew. In 1880 Hebrew was not a conversational language. In 1903 there were perhaps a few hundred Hebrew speakers. Within fifteen years more than 30,000 Jews in Palestine claimed Hebrew as their native language.[7]

While religion and language are important elements of culture their role in economic life is controversial. Are similar abrupt changes in economic culture possible? Before asserting that it is absurd to imagine that Nigerians could enrich themselves by replacing their own culture with Japanese culture, we might ask how Japanese culture came to be what it is. In 1868 Japan had a culture, technology, and standard of living similar to medieval Europe. By 1904 when Japan shocked the world by defeating a major Western military power it had transformed itself into a modern industrial state. The stunning transformation to a Western culture during the roughly 40 years of the Meiji

[5] In Acemoglu and Wolitzky [2012] this induces a cycle.
[6] See, for example, Donelly and Inglish [2010].
[7] See, for example, Bar-Adon [1972].

era affected virtually every aspect of life in Japan.[8]

The reputational theory of this paper offers a possible reconciliation between the ideas that dysfunctional norms may persist over a long period of time and the fact it is not impractical to change them. Insofar as the importance of good social norms for economic success revolves around good treatment of immigrants and foreign investors there should indeed be a reputational effect. A region that has a reputation for poor treatment of foreigners is unlikely to get much immigration or foreign investment and so is unlikely to have thriving urban centers of production and innovation. It then becomes the case that even if treatment of outsiders is improved nobody is likely to find out, so indeed the dysfunctional norm of cheating outsiders may be a form of reputation trap. I do not mean to argue that a reputation trap is the only reason dysfunctional cultures do not change: there are many costs associated with changing an entire culture and in many circumstances it may not be worth it.[9] Never-the-less given the enormous disparity in income between Nigeria and Japan we may well ask if the cost of change is the entire reason for its absence or whether a reputation trap may not play a role as well.

In a sense the phenomena our theory endeavors to explain is connected to the literature on poverty traps. That literature[10] is based on a different mechanism: it is based on the idea that there are increasing returns to scale in the accumulation of human capital. The remedies for this type of poverty trap are quite different than for a reputation trap. We do not have a great deal of evidence about the relative importance of human and social capital, but we do have the estimates of Dell and Querubin [2018] that only about one third of the persistence of poverty is due to human capital so there is substantial scope for a reputational mechanism.

A model that does lead to a reputational trap that of Board and Meyer-ter-Vehn [2013]. In that model there is a continuous reputation variable and in their bad news case if reputation falls below a threshold the firm gives up and shirks. The mechanism, however, is quite different than here: the delay that discourages effort when there is a bad reputation arises not for informational reasons but because of a lag between investment and the improvement of product quality. In that sense the model is more akin to poverty trap models than to the "pure" reputational trap analyzed here. The type of endogenous delay brought about by observational asymmetry has been studied by Ordonez [2007]. That paper is oriented towards a different set of issues than discussed here: it examines how over and under investment in signal acquisition depends on reputation. The mechanism here is also quite different than the "bad reputation" mechanism in Ely and Valimaki 2003 and Ely, Fudenberg and Levine [2008] where unraveling occurs because of a temptation to take a bad action to preserve a good reputation: in that case there is no good equilibrium at all.

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[8] See, for example, Jansen [2002].

[9] See, for example, Dutta, Levine and Modica [2019].

[10] See, for example, Azariadis and Drazen [1999].

Like the results here much recent work on reputation has focused on finding equilibria rather than computing bounds. However, unlike here, mixed strategies have played a key role. Mathevet, Pearce and Stachetti [2019] examine an information design problem where the behavioral type mixes. In Phelan [2006] it is the normal type that mixes. This leads to trust that is only gradually regained, but it does not lead to a reputation trap.

The reputational ideas here are also related to the literature on self-confirming equilibrium.[11] In that literature also a trap can arise because of the difficulty of drawing inferences about events that are rarely seen. Here we incorporate that idea into a model of rational Bayesian learning with imperfect observability and the uniqueness of equilibrium in our model enables us to draw sharp results.

I can summarize the main results of the paper by indicating the types of equilibria that occur as the cost of honesty is lowered. When the cost is very high there is a unique equilibrium in which the normal type is dishonest. There is then an intermediate range of costs in which there is a unique equilibrium that is a reputation trap. This is the crucial new finding in this paper. For lower costs there is a unique pure strategy equilibrium and two types of mixed strategy equilibria. Depending on the discount factor and rate at which information spreads the mixed strategy equilibrium may be a reputation trap, or it may be that there is a range of costs for which there is a unique equilibrium in which the normal type always invests. Regardless, once cost is low enough equilibria in which the normal type always invests co-exist with mixed equilibria. As the mixed equilibria do not have a reputation trap this simply confirms that if the cost of honesty is low enough there is no reputation trap while for intermediate levels of cost there is only a reputation trap.

2. The Model

A dynamic game is played between overlapping generations of finitely lived players. There are two player roles: player 1 is a long-run player who lives many periods and player 2 represents a mass of short-run players who live a single period. Each period $t = 1, 2, \ldots$ a stage game is played. In the stage game long-run player must first choose whether or not to make a costly investment. Let $a_1 \in \{0, 1\}$ denote the decision of the long-run player with 1 meaning to invest and the cost being ca_1 where $0 < c < 1$. The short-run player moves second and without observing the investment choice of the long-run player[12] decides whether to enter $a_2 = 1$ or stay out $a_2 = 0$. The short-run player receives utility 0 for staying out, utility -1 for entering when no investment has been made and utility $V > 0$ for entering when the investment has been made. There are three privately known types $\tau \in \{b, n, g\}$ of long-run player where g means "good" (a beneficial event), b means "bad" (an adverse event), and n means "normal." Player type is fixed during the lifetime of the player. The

[11] See, for example, Fudenberg and Levine [1993] and Sargent, Williams and Zha [2006].

[12] Meaning the game is simultaneous move.

good and bad types are behavioral types: the good type always invests and the bad type never invests. The stage game payoff of the normal type is given by $a_2 - ca_1$. Players care only about expected average utility during their lifetime.

The life of a long-run player is stochastic: with probability δ the player continues for another period, and with probability $1-\delta$ is replaced. This replacement is not observed by the short-run player. When a long-run player is replaced the type may change. The probability type τ is replaced by a type $\sigma \neq \tau$ is $Q_{\tau\sigma}\epsilon/(1-\delta)$ where $Q_{\tau\sigma} > 0$. Note that the scaling by $1-\delta$ implies that $1/\epsilon$ is a measure of the number of long-run player lifetimes before a type transition. We are interested in the case in which types are persistent - that is, in which ϵ is small.

At the beginning of each period a public signal z of what occurred in the previous period is observed and takes on one of three values: $1, 0, N$. If entry took place last period the signal is equal to last-period long-run player investment. If the short-run player stayed out last period then with probability $1 \geq \pi > 0$ the signal is equal to last period long-run player investment and with probability $1-\pi$ the signal is N. Here we are to think of "1" as a good signal (investment was observed), "0" as a bad signal (it was observed that there was no investment) and "N" as no signal. The key feature of the information technology is that when the short-run player stays out less information is generated about the behavior of the long-run player.

The game begins with an initial draw of the public signal $z(1)$ and private type $\tau(1)$ from the common knowledge distribution $\mu_{z\tau}(1)$.

Players are only aware of events that occur during their lifetime. The long-run player also knows their own generation T. Let h denote a finite history for a long-run player. A strategy for the normal type of long-run player is a choice of investment probability $\alpha_1(h, t, T)$ as a function of privately known history, calendar time, and generation T. A strategy for the short-run player is a probability of entering $\alpha_1(z, t)$ as a function of the beginning of period signal and calendar time.

We study Nash equilibria of this game.

Throughout the paper we will assume *generic cost* in the sense that

$$c \notin \left\{ \delta, \frac{\delta}{2-\pi}, \frac{\delta\pi}{1-\delta+\delta\pi}, \frac{\delta\pi(\pi-\delta\pi)}{(1-\delta\pi)(1-\delta)) + \delta\pi(\pi-\delta\pi)} \right\}.$$

Short-run Player Beliefs and Time Invariant Equilibrium

If players know calendar time, as indicated in the introduction, they can use this information to coordinate their play in an implausible way. Hence we wish to assume that short-run player strategies and beliefs are independent of calendar time.[13] Notice that this same assumption is implicit in the definition of

[13]See Clark, Fudenberg and Wolitzky [2019] for the consequences of a similar information restriction in an overlapping generations setting.

a Markov equilibrium, but is weaker since long-run player strategies may depend on the entire lifetime history of events as well as generation and calendar time.

For brevity all references to a decision problem of the long-run player should be understood to refer to the normal type. A strategy for a short-run player is a now a time invariant probability of entering $\alpha_2(z) \in [0,1]$ as a function of the beginning of period signal. Given such a strategy the normal type faces a well-posed Markov decision problem. It depends only on the probability α_2 with which the short-run player enters. Let $V(\alpha_2)$ denote the corresponding expected average value of utility. First period utility is $\alpha_2 - ca_1$. With probability δ the game continues and the probability of the next signal is $P(z'|z,a_1)$ where $P(1|z,1) = P(0|z,0) = \alpha_2(z)+(1-\alpha_2(z))\pi$ and $P(N|z,a_1) = (1-\alpha_2(z))(1-\pi)$. Hence the Bellman equation is

$$V(\alpha_2) = \max_{a_1}(1-\delta)\left[\alpha_2 - ca_1\right] + \delta \sum_{z'} P(z'|z,a_1)V(\alpha_2(z')).$$

The set of best responses, for the normal type, then, is determined entirely by the current state through $\alpha_2(z)$. Hence at time t with signal z_t any best response of the normal type $\alpha_1(y_t,t,T_t)$ must lie in this set. Time invariant beliefs of the short-run player about the investment probability of the normal type, which we denote by $\alpha_1(z)$, are then a weighted average of the best responses $\alpha_1(y_t,t,T_t)$ - and so must also be a best response and lie in this set.

Prior to observing the signal z_t the short-run player at time t has beliefs about the joint distribution $\mu_{z\tau}(t)$ from which the signal and type of the long-run player are drawn. After observing z_t short-run player beliefs about long-run player type are given by the conditional probability $\mu_{\tau|z_t}(t)$. This together with beliefs about the normal type investment $\alpha_1(z_t)$ determines $\mu^1(z_t,t)$ the overall beliefs about the probability of long-run player investment. The short-run player strategy $\alpha_2(z_t)$ must then be a best response to those beliefs.

The evolution of $\mu_{z\tau}(t)$ depends upon the initial condition $\mu_{z\tau}(1)$ and the beliefs of the short-run player about the probabilities with which earlier normal-type long-run and short-run players chose actions $\alpha_1(z)$, $\alpha_2(z)$. It does not depend on the actual choice of those actions or the earlier signals, none of which are observed. In particular no action or deviation by the long-run player has any effect on the evolution of $\mu_{z\tau}(t)$. Letting $\overrightarrow{\mu}(t)$ denote the vector with components $\mu_{z\tau}(t)$ the law of motion is $\overrightarrow{\mu}(t+1) = \mathbf{A}\overrightarrow{\mu}(t)$ where $\mathbf{A}$ is a Markov transition matrix the coefficients of which are determined by $\alpha_1(z), \alpha_2(z)$ and π, Q, ϵ.[14] To have an equilibrium with time invariant beliefs it must be that $\overrightarrow{\mu}(t+1) = \overrightarrow{\mu}(t)$ and this is true if and only if the initial condition $\mu_{z\tau}(1)$ is a stationary distribution of $\mathbf{A}$. For time invariance we cannot have arbitrary initial short-run player beliefs $\mu_{zt}(1)$, but only initial beliefs that are consistent with the strategies of the players and the passage of time.

We take our object of study, then, to be *time invariant equilibrium.* This

[14]This is computed in the Appendix.

is a Nash equilibrium in which the initial beliefs of the short-run players are determined endogenously to be the stationary distribution that arises from the equilibrium strategies. It is conveniently described as a triple $(\alpha_1(z), \alpha_2(z), \mu_{z\tau})$ where $\alpha_1(z)$ and $\mu_{z\tau}$ are time invariant beliefs of the short-run player and $\alpha_2(z)$ is the strategy of the short-run players. The conditions for equilibrium are that $\alpha_1(z)$ is a solution to the Markov decision problem induced by the short-run player strategy $\alpha_2(z)$, that $\mu_{z\tau}$ is a stationary distribution of the Markov transition matrix $\mathbf{A}$ determined by $\alpha_1(z), \alpha_2(z)$, and Q, ϵ, and that $\alpha_2(z)$ is a best response to beliefs about long-run player action $\mu^1(z)$ determined from $\alpha_1(z), \mu_{z\tau}$.

Let $z(y)$ be the most recently observed signal by the long-run player in the history y. We may conveniently summarize the discussion:

Theorem 1. *If $(\alpha_1(z), \alpha_2(z), \mu_{z\tau})$ is a time invariant equilibrium then the strategies $\alpha_1(y,t,T) = \alpha_1(z(y)), \alpha_2(z,t) = \alpha_2(z)$ are a Nash equilibrium with respect to the initial condition $\mu_{z\tau}(1) = \mu_{z\tau}$. Conversely if $\alpha_1(y,t,T), \alpha_2(z,t)$ is a Nash equilibrium that satisfies the time invariant short-run player condition that the short-run player equilibrium beliefs $\alpha_1(z,t) = \alpha_1(z), \mu_{z\tau}(t) = \mu_{z\tau}$ and equilibrium strategy $\alpha_2(z,t) = \alpha_2(z)$ then $(\alpha_1(z), \alpha_2(z), \mu_{z\tau})$ is a time invariant equilibrium.*

Hereafter by equilibrium we mean time invariant equilibrium.

3. Short-run Pure Equilibria

We first study short-run pure equilibria in which the short-run player's equilibrium strategy is pure. The different pure strategy equilibria are characterized in the following Theorem. In reading the theorem, note that $1 - \delta + \delta\pi$ is a weighted average of 1 and π so is strictly greater than π.

Theorem 2. *For given V, Q there exists an $\underline{\epsilon} > 0$ such that for $\underline{\epsilon} \min\{\pi, 1-\pi\} > \epsilon > 0$ there is a unique short-run pure equilibrium. It is a strict Nash equilibrium, and in particular the long-run player also uses a pure strategy. The short-run player enters only on the good signal. There are three mutually exclusive types of equilibria depending on c each corresponding to a different normal type long-run player pure strategy:*

i. [bad] if $c > \delta$ the normal type never invests

ii. [trap] if

$$\delta > c > \frac{\pi}{1 - \delta + \delta\pi}$$

the normal type invests only on the good signal

iii. [good] if

$$\frac{\pi}{1 - \delta + \delta\pi} > c$$

the normal type always invests.

Note that the boundary cases are ruled out by the generic cost assumption and that (at least) both types of equilibrium exist in the boundary cases.

The proof is outlined below with the detailed computations in the Appendix. The equilibrium itself is relatively intuitive. The assumption that ϵ is small means that types are highly persistent so the short-run player does not put much weight on the possibility of the type changing. Given the possible strategies of the long-run player the signal 0 indicates either a bad type or a normal type who will not invest if entry is not anticipated. Hence it makes sense for the short-run player not to enter in the face of bad signal. Similarly the signal 1 indicates either a good type or a normal type who will invest if entry is anticipated, so it makes sense for the short-run player to enter in the face of a good signal.

More subtle is the inference of the short-run player when the signal N is observed. The short-run player can infer that the previous short-run player chose not to enter - hence must have received the bad signal or was in the same boat with the signal N. As a result while less decisive than the signal 0 the signal N also indicates past bad behavior by the long-run player, so staying out is a good idea.

For the long-run player the choice is whether to invest when entry is anticipated and when it is not. The difference between the two cases lies in the probability that investment results in a good reputation which we may denote by $p = 1$ when entry is anticipated and $p = \pi$ when it is not. It is useful to consider the problem for general values of p: when the cost c is incurred there is a probability p of successfully establishing a good reputation and gaining $1 - c$ in the future and probability $1 - p$ of failing to establish a good reputation and starting over again. Here the expected average present value of the gain from investment is $\Gamma = -(1-\delta)c + \delta p(1-c) + \delta(1-p)\Gamma$ or

$$\Gamma = \frac{\delta p(1-c) - (1-\delta)c}{1 - \delta(1-p)}.$$

If this is negative, that is $\delta p(1-c) < (1-\delta)c$, then it is best not to invest and conversely. Take first the case where information is revealed immediately, that is $p = 1$. This is the situation most conducive to investment. The condition for not wishing to invest is $c > \delta$ so when this is the case there will be no investment. This is a standard case, corresponding to part (i) of the Theorem in which the long-run player is impatient and does not find it worthwhile to give up c for a future gain of $1 - c$. In this case investment will only take place only occasionally during beneficial events when the good type invests for non-reputational reasons.

When $c < \delta$ it is worth it to maintain a reputation when the short-run player enters as indeed in this case $p = 1$. The remaining question is whether it is also worth it to invest when the short-run player does not enter. In this case $p = \pi$, and the condition for investment is that given in (ii) and (iii). If c is very small then it is worth investing even when the short-run player does not enter. This good equilibrium corresponds to the "usual" reputational case, for example in Kreps and Wilson [1982], Milgrom and Roberts [1982], Fudenberg and Levine [1989], Fudenberg and Levine [1992] or Mailath and Samuelson [2001]. Here the long-run player is always is willing to invest. Occasionally an adverse event occurs and the bad type does not invest regardless of reputational consequences so investment does not take place until another normal or good type arrives.

The new and the interesting case is the trap equilibrium in case (ii) where $\delta > c$ so the cost of investment is low enough to maintain a reputation, but $c > \pi/(1-\delta+\delta\pi)$ so it is not worth it to try to acquire a reputation. Here we have strong history dependence. Depending on the history a normal type will be in one of two very different situations. A normal type that follows a history of good signals, will invest, have a good reputation and have a wealthy and satisfactory life with an income of $1-c$. A normal type that has the ill-luck to follow a history in which the last signal was bad or there was no signal will not invest, will have a (deservedly) bad reputation, and have an impoverished life with an income of 0. This we may think of as a reputational trap. The only difference between these normal types is an event that took place in the far distant past: did the last behavioral type correspond to an adverse or beneficial event? Looked at another way, adverse and beneficial events, rare as they are, cast a very long shadow. After a beneficial event there will be many lives of prosperous normal types - indeed until an adverse event occurs. Contrariwise, following an adverse event normal types will be mired in the reputation trap until they are fortunate enough to have a beneficial event. Hence, for example, an outside threat that causes people to pull together (a beneficial event) may have very long-term consequences indeed.

Observe that $\pi/(1-\delta+\delta\pi)$ is increasing in π so as π increases and news spreads quickly the range of costs for the reputation trap diminishes and we are more likely to see the "usual" good reputation case.

The Role of Behavioral Types

To better understand the role of behavioral types, consider their absence. As usual the static Nash equilibrium - always stay out and never invest - is an equilibrium. In case (i) of Theorem 2 this is the only equilibrium. For higher discount factors both the case (ii) and case (iii) strategies are Nash equilibria, although the only one that is subgame perfect is the case (ii) equilibrium in case (ii). In the usual way the presence of good types eliminates the static Nash equilibrium once the discount factor is high enough. The bad types, however, are key in selecting between the (ii) and (iii) case equilibria. The presence of behavioral types insures that the ergodic distribution is unique and that all signals (except possibly N) are present - so acts somewhat like trembles. The non-subgame perfect case (iii) equilibrium is eliminated in case (ii) and case (ii) equilibrium in case (iii) because play must be optimal following a signal of no investment. Most striking is the case (iii) equilibrium in case (iii). Despite the fact that the normal types always invest it is optimal for the short-run player to stay out on a signal of no investment: this is because such a signal indicates a bad type.

Outline of the Proof

The proof of the main theorem involves the interplay between the strategy of the long-run player and the beliefs of the short-run player. The detailed calculations are given in the Appendix through a series of Lemmas. Lemma 1 analyzes the optimum of the long-run player. It shows that regardless of the strategy of the short-run player the long-run player must invest when entry is is anticipated if she is willing to do so when entry is not anticipated. It shows in addition that unless the short-run player enters on the good signal and stays out on the bad signal the long-run player should never invest. This information is subsequently used to rule out many combinations of long-run and short-run player strategies.

The next series of steps are to characterize the ergodic beliefs of the short-run player about the long-run player. Lemma 2 examines the marginal ergodic beliefs of the short-run player about the type of long-run player. As these transition probabilities are exogenous it is straightforward to show that these beliefs do not depend on ϵ and are bounded away from zero.

The key to showing that the unique equilibrium strategy of the short-run player is to enter only on a good signal is to characterize the ergodic beliefs of the short-run player about the type of long-run player conditional on the signal. Let $\overline{B}$ be the probability of the investment that makes the short-run player indifferent to entering, that is, $\overline{B}V = (1 - \overline{B})$. Recall that $\mu^1(z)$ is the ergodic belief of the short-run player about the probability that the long-run player will invest. If $\mu^1(z) > \overline{B}$ it is strictly optimal to enter, and if it is less than this, strictly optimal to stay out. If we can show that

$$\mu^1(1) \geq 1 - K\frac{\epsilon}{\min\{\pi, 1-\pi\}}$$

and

$$\mu^1(0), \mu^1(N) \leq K\frac{\epsilon}{\min\{\pi, 1-\pi\}}$$

for some positive constant K depending only on Q then it follows that for

$$K\frac{\epsilon}{\min\{\pi, 1-\pi\}} < \min\{\overline{B}, 1-\overline{B}\}$$

it is strictly optimal for the short-run player to stay out on a bad or no signal and to enter on a good signal. This then gives the main theorem with $\underline{\epsilon} = \min\{\overline{B}, 1-\overline{B}\}/K$.

The derivation of the bounds requires several steps. Lemma 3 shows that to a good approximation the beliefs of the short-run player about the type of long-run player are the same at the beginning of a period where the type may have changed as they were at the end of the previous period. This enables us to compute approximate conditional beliefs about types and signals from the simpler problem in which types are persistent. We then want to apply Bayes law to compute the probability of types conditional on signals. To implement this we need to know a lower bound on the marginal probability of the signals: in the case of the good and bad signal this follows from the fact that the good and bad types are playing the good and bad action; the crucial case of no signal is addressed in Lemma 4 using ergodic calculations simplified by Lemma 3.

Lemma 5 then uses Bayes law for the special case in which the long-run player takes an action independent of signal (as is the case for the behavioral types).

At this point there are three possible strategies for the long-run player and eight for the short-run. It is now possible to check each of the twenty four combinations to find the ergodic beliefs and show that the only best response for the short-run player to a best response of the long-run player is to enter on a good signal and stay out for all others. Fortunately many combinations can be checked at once. This is done in Proposition 1 using the previously established bounds and partial characterization of optimal strategies.

Finally, now that we know the unique strategy of the short-run player, we must calculate the best response of the long-run player: this is the computation with Γ above.

4. Mixed Equilibria

Our first goal is to establish a sufficient condition for the pure strategy equilibrium to be the only equilibrium.

Theorem 3. *For given V, Q there exists an $\underline{\epsilon} > 0$ such that for $\underline{\epsilon}\pi^2(1-\pi) > \epsilon > 0$ and*

$$c \geq \delta \frac{1}{1+\delta(1-\pi)}.$$

all equilibria are in pure strategies.

Note that this theorem requires smaller ϵ than Theorem 2. Together with that theorem it forms the main result of this paper.

The key question is how the condition for only pure strategy equilibria in Theorem 3 overlaps with the three cases in Theorem 2. We state the comparison as a corollary:

Corollary 1. *For given V, Q there exists an $\underline{\epsilon} > 0$ such that for $\underline{\epsilon}\pi^2(1-\pi) > \epsilon > 0$ and*

i. [bad] If

$$c > \delta$$

then there is a unique equilibrium and there is no investment by the normal type.

ii. [trap] If

$$\delta > c > \delta \max\left\{\frac{\pi}{1-\delta+\delta\pi}, \frac{1}{1+\delta(1-\pi)}\right\}$$

there is a unique equilibrium and it is a reputation trap.

iii. [good] If $\pi > (1-\delta)/\delta$ and

$$\delta\frac{\pi}{1-\delta+\delta\pi} > c > \delta\frac{1}{1+\delta(1-\pi)}$$

there is a unique equilibrium and there is always investment by the normal type.

Case (i) is the not surprising result that when $c > \delta$ there is a unique bad equilibrium..

Case (ii) is the important result: it says that there is a non-trivial intermediate range of c for which there is a reputation trap. Moreover, it shows that as π is increased the range for which the reputation trap is the unique equilibrium is reduced. This reinforces our earlier discussion of the reputation trap by showing when it is the unique equilibrium.

The final result gives conditions under which there is an additional lower range of c for which there is a good equilibrium.

Proof. Case (i) follows from $1 + \delta(1 - \pi) > 1$. Case (ii) follows from $\pi/(1 - \delta + \delta\pi) < 1$. Finally, we may compute that $\pi > (1 - \delta)/\delta$ is equivalent to

$$\frac{1}{1 + \delta(1 - \pi)} < \frac{\pi}{1 - \delta + \delta\pi}$$

in which case the range of unique equilibrium extends into case (iii). □

Intuition of the Main Result

The important result is that there is a range of c for which there is a reputation trap but no mixed strategy equilibrium. Why must this be the case? The reason is that the equilibrium short-run player pure strategy of staying out on a bad or no signal $z \in \{0, N\}$and entering on a good signal $z = 1$ provides the greatest incentive for the normal type to invest. If $c > \delta$ this is not enough, so weakening the incentive to invest by mixing does not help and the only equilibrium is the one in which the normal type never invests.

To understand why the short-run player strategy must remain pure even for $c < \delta$ (but not too small) consider that at $c = \delta$ the normal type strictly prefers to not to invest on a bad or no signal and is indifferent to investment on a good signal. When c is lowered slightly the normal type now strictly prefers to invest on a good signal, while of course the strict preference on bad and no signals remain. Can there be an equilibrium in which the short-run player mixes only "a little?" That cannot happen on a bad or no signal since to get the short-run player to mix the normal type would have to mix "a lot" and this in turn would require the short-run player to mix "a lot."

What about the good signal? Here with c a little less than δ "a little" mixing by the short-run player gets the normal type back to indifference. Without types this can be an equilibrium - but not with types. The reason is tied to the ergodic distribution of types and signals. With the normal type not investing on a bad or no signal once those states are reached the normal type will no longer get the good signal. With the short-run player mixing on the good signal there is a positive probability that the normal type will get no signal: this "drains" the normal types from the good signal so that in the ergodic distribution of types and signals conditional on a good signal it is extremely likely the short-run player is facing a good type. Consequently, the short-run player will not mix on a good signal - rather the short-run player will enter for certain.

The conclusion is that mixed strategy equilibria require the short-run player to mix "a lot." Formally it is shown in Lemma 14 that in any mixed equilibrium the short-run player must be at least as likely to enter on no signal as on a good signal. This provides substantially less incentive for the normal type to invest than the short-run player equilibrium pure strategy in which the short-run player is a lot less likely to enter on no signal than on a good signal. Hence the value of c that is low enough to provide adequate incentive for investment is higher for a pure strategy equilibrium than for any mixed strategy equilibrium.

Two Types of Mixed Equilibrium

We turn now to a converse of Theorem 3: that is when

$$c < \delta \frac{1}{1 + \delta(1 - \pi)}$$

are there equilibria that are not pure? Intuitively this cannot be the case for all Q. If there are very few normal types then basically the short-run player ignores them and plays a best response to the behavioral types - which is to say the pure strategy of staying out on a bad or no signal and entering on a good signal. This we know leads the normal type to best-respond with a pure strategy as given in Theorem 2. Proposition 2 in the Appendix gives a precise result: it shows if there are enough good types there is necessarily a pure strategy equilibrium. This is not terribly interesting in itself: the case of interest is when they are many normal types, but it does show that there is no converse to Theorem 3 without an assumption on Q. Hence we investigate the interesting case of many normal types.

In addition to showing that there are mixed equilibria, we can say what they look like. There are two types, *single mixing* and *double mixing*. In both types of equilibrium in the bad state $z = 0$ there is no investment and the short-run player stays out: $\alpha_1(0) = 0$, $\alpha_2(0) = 0$. In the good state $z = 1$ both players strictly mix: $0 < \alpha_1(1) < 1$, $0 < \alpha_2(1) < 1$. In the single mixing equilibrium this is the only mixing: in the state $z = N$ the normal type invests and the short-run player enters $\alpha_1(N) = 1, \alpha_2(N) = 1$. In the double mixing case equilibrium mixing takes place also at $z = N$: the short-run player mixes exactly as in the state $z = 1$,that is $\alpha_2(N) = \alpha_2(1)$, while normal type invests with a positive probability $\alpha_1(N) > 0$.

To state a precise result and also be clear about the order of limits, it is useful to define the notion of a *fundamental bound*. This is a number that may depend on the fundamentals of the game π, V, δ, c but not on the type dynamics Q, ϵ. Recall that $\overline{B}$ is the probability of investment that makes the short-run player indifferent to entry. The main result about mixing is then Proposition 4 which we restate here as a theorem:

Theorem 4. *There exists fundamental bounds $\overline{\mu} < 1$ and $\underline{\epsilon} > 0$ such that for* *any Q with $\mu_n \geq \overline{\mu}$ if $\epsilon\underline{\mu}\pi^2(1-\pi) > \epsilon > 0$ and*

$$c < \delta\frac{1}{1+\delta(1-\pi)}$$

there is at least one single-mixing and one double-mixing equilibrium and no other type of mixed equilibrium. For each type of equilibrium there is a unique value of $\alpha_2(1)$. Moreover, for $z = 1$ in the single mixing case and $z \in \{N, 1\}$ in the double-mixing case the equilibrium value(s) of $\alpha_1(z)$ satisfies

$$|\alpha_1(z) - \overline{B}| \leq \frac{1-\mu_n}{1-\overline{\mu}}.$$

Note that we do not guarantee a unique equilibrium of each type, but show that if there are enough normal types then all equilibria of a given type are similar and the mixing by the long-run normal type is approximately the value that makes the short-run player indifferent. The reason this is only approximate is because the short-run player also faces an endogenous number of good and bad types who are either investing or not.

How do the mixed equilibria differ from the pure equilibrium? Roughly speaking we can describe the pure equilibria as having three properties: the signal is informative for the short-run player, reputation is valuable, and the normal type of long-run player remains stuck in either a good or bad situation. The mixed equilibria are quite different: the signal is uninformative for the short-run player, reputation is not valuable, and the normal type of long-run player transitions back and forth between all the states.

Specifically, with the mixed equilibrium we have the following situation. In every state the short-run player is facing mostly normal types. The normal type, starting in state $z = 0$ will eventually have some luck, the short-run player will not observe the long-run player, and the state will move to N. Here the normal type invests with positive probability and the short-run player observes this with positive probability so there is a chance of getting to the state $z = 1$. Once there both players are mixing, so there is a chance of moving to either state $z = 0$ or state $z = N$. Indeed, the only transitions that are not seen are moving directly from $z = 0$ to $z = 1$ and in the single mixing case moving directly from $z = N$ to $z = 0$. The normal type transitions back and forth between all the states. Because of this mixing the behavioral types play no role in the inferences of the short-run player. This is similar to the cheap talk literature:[15] the mixing of the long-run player effectively jams the signal of the behavioral types, and reputation plays no role in equilibrium. These equilibria also have the property that $\alpha_2(N) \geq \alpha_2(1)$: the short-run player is no more likely to enter when there is a favorable signal than when there is no signal. This represents a precise sense in which the "signal is jammed."

Finally, we emphasize that for very low c there are always signal jamming

[15] See, for example, Crawford and Sobel [1982].

equilibria: low c does not guarantee a good equilibrium.

Welfare

Is a mixed equilibrium good or bad for the long-run player? This is irrelevant in the bad equilibrium case where $c > \delta$ as there is no mixed equilibrium there. If $\pi < (1-\delta)/\delta$ and

$$\delta \frac{\pi}{1-\delta+\delta\pi} < c < \delta \frac{1}{1+\delta(1-\pi)}$$

then there is both a trap equilibrium and mixed equilibrium. The mixed equilibrium is clearly good for a long-run normal type who is trapped with no reputation - that type gets 0 while receives a positive payoff in the mixed equilibria. In this sense signal jamming is potentially good because it can alleviate a reputation trap.

On the other hand, a long-run normal type with a good reputation gets $1-c$. The next result shows that in this case a double-mixing equilibrium is unambiguously bad: expected average present value starting in the good state is strictly less.

Theorem 5. *In a double mixing equilibrium*

$$V(\alpha_2(1)) < \frac{1-\delta\pi}{1+\delta(1-\pi)} \leq 1-c.$$

Proof. From Lemma 15 in the Appendix

$$V(\alpha_2(1)) = \frac{(1-\delta\pi)\alpha_2(1)}{1+\delta(1-\pi)\alpha_2(1)}$$

which is strictly increasing in $\alpha_2(1)$, so the first bound follows from $\alpha_2(1) < 1$. The final inequality is a restatement of the condition for the existence of a double mixing equilibrium from Theorem 4. □

This has the following additional consequence. As $\delta \to 1$ regardless of initial condition utility in the good equilibrium approaches $1-c$. On the other hand the Theorem shows that $\limsup V(\alpha_2(1))$ is bounded above by $(1-\pi)/(2-\pi)$ which does not depend upon c. Hence for small enough c starting in the good state the normal long-run player does strictly worse in the double-mixing equilibrium than in the always invest equilibrium even as $\delta \to 1$. This result appears quite different than the long memory case analyzed in Fudenberg and Levine [1989] and Ekmekci, Gossner and Wilson [2012].

To understand why this is, observe that with sufficiently long memory by the short-run player the long-run player can foil a signal jamming equilibrium: if the long-run player persists in investing Fudenberg and Levine [1992] show that when there is a good type the short-run player must come to believe that the long-run player will invest. To understand how the conflict between the conclusions for $\delta \to 1$ arises, observe that for any fixed length of time the

Fudenberg and Levine [1992] bound requires the prior probability of the good type to be sufficiently high. Here the length of time is indeed fixed - the long-run player has only one period to convince the short-run player that there will be investment. Hence, as Proposition 2 in the Appendix shows, and as the Fudenberg and Levine [1992] result suggests, signal jamming is ruled out if the prior probability of the good type is sufficiently high. Hence the result here that equilibrium payoffs remain bounded away from the Stackelberg payoff of $1 - c$ when the probability of the good type is too low is an example confirming that the Fudenberg and Levine [1992] bound must depend on the strength of prior belief in the good type.

5. Conclusion

We have shown that for a non-trivial intermediate range of investment costs there is a unique equilibrium and it is a reputation trap. If this reputation trap is real we should ask the public policy question of how to get out of it. For example, if Southern Italy is caught in a reputation trap, what might the central government of Italy or the EU do to help? One possibility is to subsidize the cost of investment: if the cost c is low enough then investment even with the bad signal will be profitable and - eventually - the trap will be escaped. Welfare analysis of the model, however, indicates that this is probably not a good idea. The long-run player already has the possibility of making the investment and finds it not worth while; if the money designated for an investment subsidy was instead given to the long-run player the long-run player would choose not to spend it on investment - and would be strictly better off.

The model, however, points to another possible direction: if π could be increased it would be much easier to escape the reputation trap.[16] Here an outside agency might have an advantage over the long-run agent having, perhaps, greater influence on outsiders and information flow to outsiders. Large mega-sporting events such as a World Cup or the Olympics come to mind in this context. By bringing large numbers of outsiders a cultural change is publicized. Bearing in mind that these events are awarded many years before they take place there is increased incentive for institutional change. One reason cities and regions compete for these events is precisely in hopes of obtaining favorable publicity. We need to ask, however, has this ever worked as a means of escaping a reputation trap? Certainly to be effective the investment must actually take place – hence the Olympics in Athens in 2004 or in Rio in 2016 simply confirmed what everybody already believed about those cities. In this context it must also be emphasized that to be effective the increase in π must be large enough – it must cross the threshold for which it becomes profitable to invest on the bad signal.

[16] In a broader sense this can be thought of as part of a policy of bribing or subsidizing short-run players to enter. See, for example, Bose et al [2006]. Also relevant in this context is Vellodi [2019] who considers the design of reviewing systems to encourage entry in the face of queuing for firms with good reputation.

In the case of mega-sporting events there is an empirical literature and there are positive examples. Not all of this literature is relevant: much of it focuses on narrow issues such as tourism and local tax revenue, and venues with good reputations (where there should be little or no effect) are lumped in with venues with bad reputations (where there might be an effect). For a good overview of this literature see Matheson [2006]. Strikingly, there is evidence from Rose and Spiegel 2011 that mega-sporting events when they are combined with institutional change have a substantial effect on international trade. Examples include the Olympics awarded to China in 2001 combined with entering the WTO, the Olympics awarded to Italy in 1955 combined with a series of reforms culminating in joining the European Economic Community, the Japanese Olympics of 1964 combined with entry into the IMF and the OECD, the Olympics awarded to Spain in 1986 combined with entering the European Economic Community, the Korean Olympics of 1988 Games together with political liberalization, and the Mexican 1986 FIFA World Cup combined with entry into GATT. Two other non-sporting events that may have had a similar impact (but have not been studied empirically) are the World Exposition in Chicago in 1893 and the 1997 opening of the Guggenheim Museum in Bilbao giving rise to a revival of that city called in the popular press the "Bilbao effect.[17]"

[17] See for example "The Bilbao effect: how Frank Gehry's Guggenheim started a global craze," *The Guardian*, October 1.

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Appendix

For brevity and clarity only the results of lengthy computations are reported here. The interested reader can find the computations themselves in the online version of this appendix.

Problem of the long-run Player

We examine the problem of the normal type of long-run player. Recall the Bellman equation

$$V(\alpha_2) = \max_{a_1}(1-\delta)\left[\alpha_2 - ca_1\right] + \delta \sum_{z'} P(z'|z, a_1)V(\alpha_2(z')).$$

We may write this out as

$$V(\alpha_2) =$$

$$\max_{a_1}(1-\delta)\left[\alpha_2 - ca_1\right] + \delta\left[\left(\alpha_2 + (1-\alpha_2)\pi\right)V(\alpha_2(a_1)) + (1-\alpha_2)(1-\pi)V(\alpha_2(N))\right].$$

Lemma 1. *The optimum for the normal type of long-run-player depends on the state only through α_2 and one of three cases applies:*

(i) $V(\alpha_2(1)) - V(\alpha_2(0)) < c(1-\delta)/\delta$: it is strictly not optimal to invest in every state. In particular if $\alpha_2(1) = \alpha_2(0)$ this is the case.

(ii) $V(\alpha_2(1)) - V(\alpha_2(0)) > c(1-\delta)/(\delta\pi)$: it is strictly optimal to invest in every state

Defining

$$\tilde{\alpha}_2 = \frac{1-\delta}{\delta(1-\pi)\left(V(\alpha_2(1)) - V(\alpha_2(0))\right)}c - \frac{\pi}{1-\pi}$$

(iii) it is strictly optimal to invest if $\alpha_2(z) > \tilde{\alpha}_2$ and conversely. In particular the strategy $\alpha_1(0) > \alpha_1(1)$ is never optimal.

In addition

(iv) if $\alpha_2(0) = 1$ then it is strictly optimal not to invest in every state.

Finally, if the short-run player uses a pure strategy then the optimum of the long-run player is strict and pure.

Proof. The argmax is derived from:

$$\max_{a_1} -(1-\delta)ca_1 + \delta\left(\alpha_2 + (1-\alpha_2)\pi\right)V(\alpha_2(a_1)).$$

The gain to not investing is

$$G(\alpha_2) = (1-\delta)c - \delta\left(\alpha_2 + (1-\alpha_2\pi)\left[V(\alpha_2(1)) - V(\alpha_2(0))\right]\right.$$

We then solve this equation form α_2 to see when investment is and is not optimal.

Finally, we analyze best response of the long-run player when the short-run player uses a pure strategy. From (i) and (iv) if $\alpha_2(0) \geq \alpha_2(1)$ it is strictly best not to invest. That leaves only the case $\alpha_2(a_1) = a_1$, or rather two cases, depending on $\alpha_2(N)$. This is a matter of solving the Bellman equations for each case to determine the value of c (if any) there can be a tie. This are the "non-generic" values listed in the text. $\square$

Ergodic Beliefs of the Short-Run Player

Next we examine the beliefs of the short-run player. For given pure strategies of both players the signal type pairs (z, τ) are a Markov chain with transition probabilities independent of δ and depending only on ϵ, π and the strategies of the two players. Excluding the state N in case the short-run player always enters the chain is irreducible and aperiodic so it has a unique ergodic distribution $\mu_{z\tau}$. We first analyze the marginals μ_τ and μ_z.

Lemma 2. *The marginals μ_τ are independent of ϵ. Let $\underline{\mu} = \min_{\tau \neq n} \mu_\tau$. Then $\underline{\mu} > 0$, $\mu_0, \mu_1 \geq \pi\underline{\mu}$, if $\alpha_2(0) = \alpha_2(1) = 1$ then $\mu_N = 0$, otherwise if the short-run player plays a pure strategy then $\mu_N \geq (1-\pi)\underline{\mu}$.*

Proof. The type transitions are independent of the signals, so we analyze those first. For $\epsilon > 0$ we have $\mu_\tau > 0$ since every type transition has positive probability. This ergodic distribution is the unique fixed point of the 3×3 transition matrix A, which is to say given by the intersection of the null space of $I - A$ with the unit simplex. Since $A = I + Q\epsilon$ it follows that it is given by the intersection of the null space of $Q\epsilon$ with the unit simplex. As the null space of $Q\epsilon$ is independent of ϵ the marginals μ_τ are independent of ϵ as well.

For the signals we have $\mu_1 \geq \pi\mu_g$ and $\mu_0 \geq \pi\mu_b$. If if $a_2(0) = a_2(1) = 1$ then the state N is transient. If $\alpha_2(1) = 0$ then $\mu_N \geq (1-\pi)\mu_g$ while if $\alpha_2(0) = 0$ then $\mu_N \geq (1-\pi)\mu_b$. $\square$

It will be convenient to normalize so that $\max(\mu_\sigma/\mu_\tau)Q_{\tau\sigma} = 1$. Next we show how the conditional probabilities $\mu_{z|\tau}$ can be computed approximately by using the ergodic conditions for $\epsilon = 0$.

Lemma 3. *When $z = N$*

$$\mu_{N|\tau} = (1-\pi)\left(\sum_y (1-\alpha_2(y))\mu_{y|\tau} + \epsilon H_{N\tau}\right)$$

when $z \neq N$

$$\mu_{z|\tau} = \sum_y 1\left((z=1)\alpha_1(\tau, y) + 1(z=0)(1-\alpha_1(\tau, y))\right)\left[\alpha_2(y) + \pi(1-\alpha_2(y))\right]\mu_{y|\tau} + \epsilon H_{z\tau}.$$

where $|H_{z\tau}| \leq 2$ for all z.

Proof. The idea is that the process for types is exogenous, so the stationary probabilities can be computed directly. This enables us to find a linear recursive relationship for the conditionals where the coefficients depend upon the strategies and the (already known) marginals over types. We then show that when ϵ is small to a good approximation we can do the computation for $\epsilon = 0$, that is, ignoring the type transitions, with the result above showing how good the approximation is for given ϵ. $\square$

To apply Bayes Law we will need to bound marginal probabilities of signals from below. The hard case is that of no signal where we must solve the equations for the conditionals simultaneously. Here we analyze the short-run pure strategy case. If the short-run player enters for both $z = 0, 1$ then no signals are unlikely as they are generated only from type transitions, so we rule that out.

Lemma 4. *Suppose $\alpha_2(a_1) = 0$ for some $a_1 \in \{0, 1\}$. Then*

$$\mu_N \geq \frac{1-\pi}{2}\left(1 - \frac{4\epsilon}{\pi}\right)\underline{\mu}.$$

Proof. Let τ be the type that plays a_1. We have

$$\mu_{a_1|\tau} = \sum_y \left[\alpha_2(y) + \pi(1-\alpha_2(y))\right]\mu_{y|\tau} + \epsilon H_{a_1\tau}$$

$$\mu_{N|\tau} = (1-\pi)\left(\sum_y (1-\alpha_2(y))\mu_{y|\tau} + \epsilon H_{N\tau}\right)$$

These imply the inequalities

$$\mu_{a_1|\tau} \geq \pi(1-\mu_{N|\tau}) + \left[\alpha_2(N) + \pi(1-\alpha_2(N))\right]\mu_{N|\tau} + \epsilon H_{a_1\tau}$$

$$\mu_{N|\tau} \geq (1-\pi)\left((1-\alpha_2(N))\mu_{N|\tau} + \mu_{a_1|\tau} + \epsilon H_{N\tau}\right).$$

Hence

$$\begin{aligned}\mu_{N|\tau} &\geq (1-\pi)\left((1-\alpha_2(N))\mu_{N|\tau} + \pi(1-\mu_{N|\tau}) + \left[\alpha_2(N) + \pi(1-\alpha_2(N))\right]\mu_{N|\tau} + \epsilon H_{N\tau} + \epsilon H_{a_1\tau}\right)\\ &\geq (1-\pi)\left(\pi + (1-\pi)\mu_{N|\tau} + \epsilon H_{N\tau} + \epsilon H_{a_1\tau}\right).\end{aligned}$$

It follows that

$$\mu_{N|\tau} \geq \frac{1-\pi}{2}\left(1 - \frac{4\epsilon}{\pi}\right).$$

The result now follows from $\mu_N \geq \mu_{N|\tau}\mu_\tau \geq \mu_{N|\tau}\underline{\mu}$. □

Finally we compute bounds on beliefs about types that play the same action independent of the signal. Here we combine bounds from the equations for the conditionals with Bayes Law.

Lemma 5. *A long-run type τ that plays the pure action a_1 regardless of the signal has*

$$\mu_{\tau|-a_1} \leq \frac{2}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

and if $\alpha_2(1) = 1$ and $\alpha_2(0) = 0$ then a type τ that plays the action 1 regardless of signal has

$$\mu_{\tau|N} \leq \frac{8}{\left(1 - 4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}}\left(\frac{\epsilon}{\pi}\right).$$

Proof. If long-run type τ plays the pure action a_1 from Lemma 3 $\mu_{-a_1|\tau} = \epsilon H_{-a_1\tau} \leq 2\epsilon$. From Lemma 2 $\mu_{-a_1} \geq \pi\underline{\mu}$ and Bayes law then implies

$$\mu_{\tau|-a_1} \leq \frac{\epsilon 2}{\pi\underline{\mu}}.$$

For the second part we have from Lemma 3

$$\mu_{N|\tau} = (1-\pi)\left(\mu_{0|\tau} + [1-\alpha_2(N)]\,\mu_{N|\tau}\right) + (1-\pi)\epsilon H_{N\tau}.$$

$$\mu_{0|\tau} = \epsilon H_{0\tau}.$$

Plugging in $\mu_{N|\tau} \leq (1-\pi)\mu_{N|\tau} + (1-\pi)\epsilon H_{0\tau} + (1-\pi)\epsilon H_{N\tau}$ so

$$\mu_{N|\tau} \leq \frac{(1-\pi)4\epsilon}{\pi}.$$

From Lemma 4

$$\mu_N \geq \frac{1-\pi}{2}\left(1-\frac{4\epsilon}{\pi}\right)\underline{\mu}.$$

Hence Bayes law implies

$$\mu_{\tau|N} \leq \frac{8\epsilon}{\pi\left(1-\frac{4\epsilon}{\pi}\right)\underline{\mu}}.$$

□

Short-Run Player Optimality

Recall that $\mu^1(z)$ is the probability of $a_1 = 1$ in state z and that $\overline{B} = 1/(V+1)$ is the critical value of $\mu^1(z)$ such that

Lemma 6. *If $\mu^1(z) > \overline{B}$ the short-run player strictly prefers to enter; if $\mu^1(z) < \overline{B}$ the short-run player strictly prefers to stay out, and if $\mu^1(z) = \overline{B}$ the short-run player is indifferent.*

We next show that it cannot be optimal for the short-run player always to enter. Set $B \equiv \underline{\mu}\min\{\pi, 1-\pi\}\min\{\overline{B}, 1-\overline{B}\}$.

Lemma 7. *For $\epsilon < (1/2)B$ always enter $a_2(z) = 1$ for all z is not an equilibrium.*

Proof. By Lemma 1 always enter implies no investment by the normal long-run player. As there are few good types at $z = 0$ we show that this forces the short-run player to stay out there so the short-run player should not in fact enter. □

Lemma 8. *For $\epsilon < (1/16)B$ the strict equilibrium response to never invest is to enter only on $z = 1$ and do so with probability 1.*

Proof. As the normal and bad types never invest the signal $z = 1$ implies a good type with high probability so the short-run player should enter there. This means that the long-run player can have the signal $z = 1, N$ only through a type transition. In particular the bad signal is dominated by normal and bad types so the short-run player should stay out. This in turn means that most of the N signals are generated by normal and bad types, so the short-run player should stay out there too. □

Lemma 9. *For $\epsilon < (1/16)B$ there is no equilibrium in which $\alpha_2(0) = 1$.*

Proof. By Lemma 1 $\alpha_2(0) = 1$ implies never invest so by Lemma 8 $\alpha_2(0) = 0$ a contradiction. □

Lemma 10. *For $\epsilon < (1/32)B$ the unique equilibrium response to always invest is to enter only on $z = 1$ and do so with probability 1.*

Proof. This is basically the opposite of Lemma 8. Now at $z = 1$ there are mainly good and normal types so it is optimal for the short-run player to enter. While at $z = 0$ there are mainly bad types so it is optimal for the short-run player to stay out. Hence no-signal is generated by bad types from $z = 0$ so it is optimal for the short-run player to stay out there too. □

Lemma 11. *If $\epsilon < (1/2)B$ and for some a_1 we have $\alpha_1(a_1) = a_1$ then $\alpha_2(a_1) = a_1$.*

Proof. If $\alpha_1(0) = 0$ then from Lemmas 3 and 2 $\mu^1(0) = \mu_{0|g}\mu_g/\mu_0 = \epsilon H_{0g}\mu_g/\mu_0 \leq 2\epsilon/(\pi\underline{\mu})$. If $\alpha_1(1) = 1$ then $1-\mu^1(1) = \mu_{1|b}\mu_b/\mu_1 = \epsilon H_{1b}\mu_b/\mu_1 \leq 2\epsilon/(\pi\underline{\mu})$. Hence for $\epsilon/\pi < \overline{B}\underline{\mu}/2$ it follows that $\alpha_2(a_1) = a_1$. □

Uniqueness of Short-Run Pure Equilibria

We define an *equilibrium response* of the short-run player to a strategy of the long-run player to be a best response to $\mu_{z\tau}$ induced by the long-run player strategy and itself.

Proposition 1. *There exists an $\underline{\epsilon} > 0$ depending only on V such that for any ϵ satisfying*

$$\underline{\epsilon} > \frac{\epsilon}{\underline{\mu}\min\{\pi, 1-\pi\}} > 0$$

in any short-run pure equilibrium the short-run player must enter on the good signal and only on the good signal. Moreover this is a strict equilibrium response.

Proof. We rule out all other possibilities:

(a) A*lways enter $a_2(z) = 1$ for all z is not an equilibrium.* By Lemma 7

(b) *The unique equilibrium response to never invest is to enter only on $z = 1$.* From Lemma 7.

(c) *A equilibrium response requires $a_2(1) = 1, a_2(0) = 0$.* Any other strategy satisfies $a_2(0) \geq a_2(1)$. From Lemma 1 this implies no investment by the long-run player. Part (b) then forces $0 = a_2(0) < a_2(1) = 1$ a contradiction.

(d) *The unique equilibrium response to always invest is to enter only on* $z = 1$. From Lemma 10.

This leaves only the strategy $\tilde{a}$ in which the long-run player plays $a_1 = 1$ on entry and $a_1 = 0$ if the short-run player stays out. As we know that $\alpha_2(1) = 1, \alpha_2(0) = 0$ there are two possibilities $\alpha_2(N) = 1$ and $\alpha_2(N) = 0$. The former is ruled out because it leads to primarily bad types at $z = N$, and the latter is a strict best response by the short-run player because there are few good types at $z = N$. □

Mixing

Recall that all of the Lemmas concerning short-run optimality hold for $\epsilon \leq B/32$ (and the remaining Lemmas do not place restrictions on ϵ) where $B = \underline{\mu} \min\{\pi, 1-\pi\} \min\{\overline{B}, 1-\overline{B}\}$. Recall also the notion of a fundamental bound: it may depend on the fundamentals of the game π, V, δ, c but not on the type dynamics Q, ϵ. Define the fundamental bound $\overline{A} \equiv \pi^2(1-\pi)\min\{\overline{B}, 1-\overline{B}\}$ and observe that if $\epsilon \leq \underline{\mu}\overline{A}/32$ then also $\epsilon \leq B/32$. We shall assume $\epsilon \leq \underline{\mu}\overline{A}/32$ hereafter.

Lemma 12. *There is no non-pure equilibrium with* $\alpha_1(1) = 1$.

Proof. By Lemma 2 $\mu_{1|b} = \epsilon H_{1b} \leq 2\epsilon$. Hence for $\epsilon < \overline{B}/2$ by Lemma 6 $\alpha_2(1) = 1$. Then by Lemma 2 $\mu_{1|n} = \mu_{1|n} + \sum_{y\in\{0,N\}} \alpha_1(y)\left[\alpha_2(y) + \pi(1-\alpha_2(y))\right]\mu_{y|n} + \epsilon H_{z\tau}$.I t follows that

$$\sum_{y\in\{0,N\}} \alpha_1(y)\mu_{y|n} \leq 2(\epsilon/\pi) \text{ so } \max_{y\in\{0,N\}} \alpha_1(y)\mu_{y|n} \leq 2(\epsilon/\pi).$$

Moreover for $z \in \{0, N\}$ we have $\mu_{z|g} = \epsilon H_{zg} \leq 2\epsilon$. Hence

$$\mu^1(0) = \frac{\mu_{0|g}\mu_g + \alpha_1(0)\mu_{0|n}\mu_n}{\mu_0} \leq 2(\epsilon/\pi)(\mu_g + \mu_n)/(\pi\underline{\mu}) \leq 2(\epsilon/\pi)/(\pi\underline{\mu}).$$

So for $\epsilon/\pi^2 < \overline{B}\underline{\mu}/2$ (this is why π^2 appears in $\overline{A}$) by Lemma 6 we have $\alpha_2(0) = 0$. This implies by Lemma 4 that

$$\mu^1(N) = \frac{\mu_{N|g}\mu_g + \alpha_1(N)\mu_{N|n}\mu_n}{\mu_N} \leq 2(\epsilon/\pi)(\mu_g + \mu_n)/\mu_N$$
$$\leq \frac{8(\epsilon/\pi)}{(1-\pi)\left(1-\frac{4\epsilon}{\pi}\right)\underline{\mu}}.$$

So when this is less than or equal $\overline{B}$ by Lemma 6 we have $\alpha_2(N) = 0$. For $\epsilon \leq \overline{A}/8$ this is

$$\frac{16\epsilon}{\pi(1-\pi)\underline{\mu}} \leq \overline{B}$$

so holds for $\epsilon < \underline{\mu}\overline{A}/16$ which was assumed. □

Lemma 13. *In any equilibrium* $\alpha_1(0) = \alpha_2(0) = 0$.

Proof. We already know this to be true in any pure equilibrium, so we may assume the equilibrium is not pure. From Lemma 11 if $\alpha_1(0) = 0$ then $\alpha_2(0) = 0$ so we may assume this is not the case, that is $\alpha_1(0) > 0$. From Lemma 12 we know that $\alpha_1(1) < 1$. It cannot be that the normal type is indifferent at both $z = 0, 1$ for then by Lemma 1 it must be that $\alpha_2(1) = \alpha_2(0) = \tilde{\alpha}_2$ so that $V_1 = V(\tilde{\alpha}) = V_0$ and that the normal type never invests in which case by Lemma 8 we would have a pure strategy equilibrium. Hence either the normal type strictly prefers not to invest at $z = 1$ and is willing to invest at $z = 0$ or the normal type is indifferent at $z = 1$ and strictly prefers to invest at $z = 0$. In either case from Lemma 1 we must have $\alpha_2(1) < \alpha_2(0)$.

The key point is that having the short-run player enter when there is no investment is kind of like winning the lottery - you get something for nothing. If that happens in the state 0 it is particularly good because you are guaranteed that you get to play again. Since $\alpha_2(1) < \alpha_2(0)$ we can write $\alpha_2(0) = \beta + (1 - \beta)\alpha_2(1)$ where $\beta > 0$ meaning that in the state $z = 0$ there is a better chance of winning the lottery. We will use this to show that $V(\alpha_2(0)) \geq V(\alpha_2(1))$ so that never invest is optimal and the equilibrium must be pure by Lemma 8. □

Lemma 14. *In any non-pure equilibrium* $0 < \alpha_2(1) < 1$, $\alpha_1(N) > 0$, *and* $\alpha_2(N) \geq \alpha_2(1)$.

Proof. First suppose that $\alpha_2(1) = 1$. Since the short-run player must be mixing and by Lemma 13 is not doing so at $z = 0$ the short-run player must be mixing at $z = N$, that is, that $0 < \alpha_2(N) < 1$. Lemma 12 implies that at $z = 1$ the normal type does not strictly prefer to invest. Since $\alpha_2(N) < \alpha_2(1)$ Lemma 1 implies that at $z = N$ normal type strictly prefers not to invest, so $\alpha_1(N) = 0$. Hence $\mu^1(N) = \mu_{N|g}\mu_g/\mu_N = \epsilon H_{0g}\mu_g/\mu_N$. As $\alpha_2(0) = 0$ by Lemma 13 it follows from Lemma 4 that

$$\mu^1(N) \leq \frac{4\epsilon}{(1-\pi)\left(1 - \frac{4\epsilon}{\pi}\right)\underline{\mu}}$$

as the RHS this is less than $\overline{B}$ by assumption we have $\alpha_2(N) = 0$ a contradiction.

Next suppose that $\alpha_2(1) = 0$. By Lemma 13 we also have $\alpha_2(0) = 0$ so by Lemma 1 the long-run player never invests. Hence $\alpha_2(1) > 0$ follows from Lemma 8, a contradiction. We have now shown strict mixing the the short-run player at $z = 1$.

Now we show that since the short-run player is strictly mixing at $z = 1$ then $\alpha_1(N) > 0$. Strict mixing by the short-run player at $z = 1$ implies from Lemma 6 $1 - \overline{B} = 1 - \mu^1(1) = \left([1 - \alpha_1(1)]\mu_{1|n}\mu_n + \mu_{1|b}\mu_b\right)/\mu_1$. From Lemma 3 and Lemma 13 if $\alpha_1(N) = 0$ we have $\mu_{1|n} \leq \alpha_1(1)\mu_{1|n} + 2\epsilon$ and $\mu_{1|b} \leq 2\epsilon$. Hence by Lemma 2 $1 - \mu^1(1) \leq 2\epsilon/(\pi\underline{\mu})$, so for $2\epsilon/(\pi\underline{\mu}) < 1 - \overline{B}$ this is a contradiction.

Since $\alpha_2(N) > 0$ the normal type weakly prefers to invest at $z = N$. If $\alpha_2(1) > \alpha_2(N)$ by Lemma 1 this implies the normal type would strictly prefer to invest at $z = 1$ contradicting Lemma 12. □

Proposition 2. *For any Q with*

$$\mu_g > \frac{\overline{B}}{\overline{B} + (1-\overline{B})(\pi/2)}$$

if $\epsilon \leq \underline{\mu}\overline{A}/32$ then all equilibria are pure.

Proof. From Bayes Law

$$\mu^1(1) \geq \mu_{g|1} = \frac{\mu_{1|g}\mu_g}{\mu_{1|g}\mu_g + \sum_{\tau \neq g} \mu_{1|\tau}\mu_\tau} \geq \frac{1}{1 + (1-\mu_g)/(\mu_{1|g}\mu_g)}.$$

From Lemma 3

$$\mu_{1|g} \geq [\alpha_2(1) + \pi(1-\alpha_2(1))]\,\mu_{1|g} + [\alpha_2(N) + \pi(1-\alpha_2(N))]\,\mu_{N|g} - 2\epsilon.$$

The same Lemma implies $\mu_{0|g} \leq 2\epsilon$, so

$$\mu_{1|g} \geq [\alpha_2(1) + \pi(1-\alpha_2(1)) - \pi]\,\mu_{1|g} + \pi - 4\epsilon \geq \pi - 4\epsilon.$$

Combining the two

$$\mu^1(1) \geq \frac{1}{1 + (1-\mu_g)/((\pi - 4\epsilon)\mu_g)}.$$

By Lemma 5 if

$$\frac{1}{1 + (1-\mu_g)/((\pi - 4\epsilon)\mu_g)} > \overline{B}$$

or equivalently

$$\mu_g > \frac{\overline{B}}{\overline{B} + (1-\overline{B})(\pi - 4\epsilon)}$$

then $\alpha_2(1) = 1$ so the result follows from Lemma 14 and the assumption that $\epsilon < \underline{\mu}\overline{A}/32 \leq \pi/2$. $\square$

Signal Jamming

Define the *auxiliary system* with respect to $0 \leq \lambda, \gamma \leq 1$ as

$$V_1 = (1-\delta)\tilde{\alpha}_2 + \delta\left[(\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi)\,V_0 + (1-\tilde{\alpha}_2)(1-\pi)V_N\right]$$

$$V_N = (1-\gamma)(\lambda - c) + \gamma V_1$$

$$V_0 = \frac{\delta(1-\pi)}{1-\delta\pi} V_N.$$

Since in a mixed equilibrium we know from Lemma 12 that $\alpha_1(1) < 1$ so that at $z = 1$ the long-run player must be willing not to invest. This system corresponds to not investing at $z = 0, 1$. From the contraction mapping fixed point theorem this has a unique solution V_1, V_N, V_0. Define the function $\underline{\Delta}(\tilde{\alpha}_2) \equiv V_1 - V_0$.

Lemma 15. *We have*

$$V_1 = \frac{\delta(1-\pi)(1-\gamma)(\lambda - c) + (1-\delta)\left[1 - \delta\pi - \delta(1-\pi)(1-\gamma)(\lambda - c)\right]\tilde{\alpha}_2}{(1 - \delta\pi - \gamma\delta(1-\pi)) + \gamma\delta(1-\pi)(1-\delta)\tilde{\alpha}_2}$$

strictly increasing in $\tilde{\alpha}_2$.

Proof. Here we simply solve the linear system and determine the sign of the derivative of V_1. □

Lemma 16. $\underline{\Delta}(\tilde{\alpha}_2)$ *is strictly increasing. There is a solution* $0 < \hat{\alpha}_2 < 1$ *to*

$$\underline{\Delta}(\tilde{\alpha}_2) = \overline{\Delta}(\tilde{\alpha}_2) \equiv \frac{1-\delta}{\delta\left(\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi\right)}c,$$

it and only if

$$c < \delta\frac{(1 - \delta\pi - \delta(1-\pi)\left[\gamma + \lambda(1-\gamma)\right])}{1 - \delta\pi - \delta^2(1-\pi)},$$

in which case it is unique.

Proof. Here solve V_0 as a function of V_1 from the system. We subtract this from V_1 and find that $\underline{\Delta}(\tilde{\alpha}_2)$ is strictly increasing in V_1. Hence we may apply Lemma 15. Since $\overline{\Delta}(\tilde{\alpha}_2)$ is decreasing there will be a unique intersection if and only if $\overline{\Delta}(0) > \underline{\Delta}(0)$ and $\overline{\Delta}(1) < \underline{\Delta}(1)$. By computation we show that the first condition is always satisfied and the second is the condition on c given as the result. □

Proposition 3. *If* $\epsilon < \underline{\mu}\pi^2(1-\pi)\min\{\overline{B}, 1-\overline{B}\}/32$ *and*

$$c \geq \delta\frac{1}{1 + \delta(1-\pi)}.$$

all equilibria are in pure strategies.

Proof. Suppose that $\alpha_1(z), \alpha_2(z)$ is a non-pure equilibrium. If the normal type is willing to invest at $z = 1$ we take $\hat{\alpha}_2 = \alpha_2(1)$. If the long-run player strictly prefers not to invest at $z = 1$ we show how to construct a $1 > \hat{\alpha}_2 > \alpha_2(1)$ for which the long-run player is indifferent at $z = 1$ and strictly prefers to invest at $z = N$. We show that $1 - c \geq V(\alpha_2(N)) \geq V(\hat{\alpha}_2)$ and use this to show that at $\hat{\alpha}_2$ we must have $\underline{\Delta}(\hat{\alpha}_2) = \overline{\Delta}(\hat{\alpha}_2)$ for $\lambda = 1$. Applying Lemma 16 then yields the desired condition. □

Role of Types

It is useful at this point to recall the notion of a single mixing and double mixing profile. In both profiles $\alpha_1(0) = 0$, $0 < \alpha_1(1) < 1$, $\alpha_2(0) = 0$, $0 < \alpha_2(1) < 1$. In a single mixing profile $\alpha_1(N) = 1, \alpha_2(N) = 1$, while in a double-mixing profile $\alpha_1(N) > 0$ and $\alpha_2(N) = \alpha_2(1)$.

Lemma 17. *There exists a fundamental bound $\overline{\mu} < 1$ such that for any Q with $\mu_n \geq \overline{\mu}$ if for $\epsilon \leq \underline{\mu}\overline{A}/32$ a non-pure equilibrium is either a single- or double-mixing profile.*

Proof. The only things not covered in Lemma 12, 13, and 14 are $\alpha_1(1) \neq 0$ and the result that $\alpha_2(N) > \alpha_2(1)$ implies $\alpha_1(N) = 1, \alpha_2(N) = 1$.

For the first result, the idea is since μ_n is large there must be many more normal types at N than good types. Since since $\alpha_2(N) > 0$ this means that $\alpha_1(N)$ cannot be too small, and this in turn implies that even though $\alpha_1(1) = 0$ there must be many more normal types at 1 then good types. If they not investing then the short-run player should stay out contradicting the fact that we already know $\alpha_2(1) > 0$.

For the second result we leverage the first to see that we must have $\alpha_1(N) = 1$. Moreover, since $\alpha_1(1) < 1$ there must be many normal types at $z = 0$, and so at $z = N$. As these are all investing, it is optimal for the short-run player to enter. □

Lemma 18. *In any single- or double-mixing profile if $\mu_n \geq 1/2$ and $\epsilon < (1 - a_2(1))(1-\pi)/12$ then*

$$\mu_{n|N} \geq 1 - \frac{1-\mu_n}{(1-\alpha_2(1))(1-\pi)/12}.$$

If in addition $\epsilon < \alpha_1(N)\pi(1-a_2(1))(1-\pi)/24$ then

$$\mu_{n|1} \geq 1 - \frac{1-\mu_n}{\alpha_1(N)\pi(1-\alpha_2(1))(1-\pi)/24}.$$

Proof. The first result says that if $\alpha_2(1)$ is less than 1 and if there are many normal types there must be many normal types at $z = N$, as they are flowing there from both $z = 0$ and $z = 1$. The second result leverages this to say that if there are many normal types at $z = N$ and $\alpha_1(N)$ is large then there must be many normal types at $z = 1$. □

The next Lemma is simply an observation:

Lemma 19. *A single mixing equilibrium corresponds to the auxiliary system with $\lambda = 1$ and $\gamma = \delta$ and a double mixing equilibrium corresponds to the auxiliary system with $\lambda = 1$ and $\gamma = 1$. In particular in a single mixing equilibrium*

$$V(\alpha_2(1)) = \frac{(1-\delta\pi)\alpha_2(1)}{1+\delta(1-\pi)\alpha_2(1)}$$

which is increasing in $\alpha_2(1)$.

Proof. In the single mixing case this is just the Bellman equation. In the double mixing case we use the fact that $V(\alpha_2(N)) = V(\alpha_2(1))$. The value $V(\tilde{\alpha}_2)$ follows from plugging into the expression for V_1 in Lemma 15; that Lemma gives the result that it is increasing. □

Proposition 4. *There exists a fundamental bound $\overline{\mu} < 1$ such that for any Q with $\mu_n \geq \overline{\mu}$ if $\epsilon \leq \underline{\mu}\overline{A}/32$ and*

$$c < \delta \frac{1}{1 + \delta(1 - \pi)}$$

there is at least one single-mixing and one double-mixing equilibrium and no other type of mixed equilibrium. In both cases the equilibrium value of $\alpha_2(1)$ is the unique solution of $\underline{\Delta}(\alpha_2(1)) = \overline{\Delta}(\alpha_2(1))$ where $\lambda = 1$ and in the single-mixing case $\gamma = \delta$ and in the double-mixing case $\gamma = 1$. Moreover, the equilibrium value of $\alpha_1(z)$ satisfies

$$|\alpha_1(z) - \overline{B}| \leq \frac{1 - \mu_n}{1 - \overline{\mu}}$$

for $z = 1$ in the single mixing case and $z \in \{N, 1\}$ in the double-mixing case.

Proof. From Lemma 17 we know there can be no other kind of equilibrium. From Lemma 16 we know that

$$c < \delta \frac{(1 - \delta\pi - \delta(1 - \pi)\,[\gamma + \lambda(1 - \gamma)])}{1 - \delta\pi - \delta^2(1 - \pi)}$$

and from Lemma 19 with $\lambda = 1$ and $\gamma = \delta$ is a necessary condition for the existence of single-mixing equilibrium and with $\lambda = 1$ and $\gamma = 1$ for the existence of a double-mixing equilibrium. When $\lambda = 1$ the RHS is independent of γ and given as the expression in the Theorem.This gives us a unique solution $0 < \tilde{\alpha}_2 < 1$ for the equilibrium value of $\alpha_2(1)$. The crucial fact is that $\tilde{a}_2$ arising from the optimization problem of the normal type is itself a fundamental bound.

We must now show the existence of an $\alpha_1(1)$ so that the short-run player is indifferent when $z = 1$ and weakly prefers to enter when $z = N$, and in the double mixing case the existence of $\alpha_1(1), \alpha_1(N)$ so that the short-run player is indifferent in both $z = N, 1$, and that any such strategic components satisfy the required bound.

Recall that $\mu^1(z)$ are the beliefs of the short-run player about the probability the long-run player will invest. This is given as $\mu^1(z) = \mu_{g|z} + \mu_{n|z}\alpha_1(z)$. Define $\tilde{A}(z, \alpha_1(z)) = \mu_1(z) - \overline{B}$. Hence the equilibrium requirement is that $\tilde{A}(1, \alpha_1(1)) = 0$ and that in the single mixing case $\tilde{A}(N, \alpha_1(N)) = 0$ and in the double-mixing case $\tilde{A}(N, 1) \geq 0$. The complication is that $\mu_{g|z}$ and $\mu_{n|z}$ for $z \in \{N, 1\}$both depend upon $\alpha_1(1)$ and $\alpha_1(N)$. As by the ergodic theorem the ergodic distribution is continuous in $\alpha_1(1)$ and $\alpha_1(N)$ so are $\tilde{A}(z, \alpha_1(z))$ and we will be able to apply fixed point argument.

Write $\tilde{A}(z, \alpha_1) = \mu_{g|z} - (1 - \mu_{n|z})\alpha_1 + \alpha_1 - \overline{B}$ and observe that $\mu_{g|z} \leq (1 - \mu_{n|z})$. Hence $\tilde{A}(z, \alpha_1) = \alpha_1 - \overline{B} + \tilde{A}_1(1 - \mu_{n|z})$ with $|\tilde{A}_1| \leq 2$.

We now apply the first bound from Lemma 18. We know that $\alpha_2(1) = \tilde{\alpha}_2$ a fundamental bound so we have $\tilde{A}(N, \alpha_1) = \alpha_1 - \overline{B} + \tilde{A}_2(1 - \mu_n)$ where $|\tilde{A}_2| \leq A_2$ and A_2 is a fundamental bound. Hence for $\alpha_1 - \overline{B} \leq -A_2(1 - \mu_n)$

we have $\tilde{A}(N, \alpha_1) < 0$. Taking $A_2(1 - \mu_n) \leq \overline{B}/2$ for $\alpha_1 \leq \overline{B}/2$ we also have $\tilde{A}(N, \alpha_1) < 0$. We may restrict attention then to the region where $\alpha_1(N) \geq \overline{B}/2$ since there can be no equilibrium outside this region.

In the region $\alpha_1(N) \geq \overline{B}/2$ we may now apply the second bound from Lemma 18 and find that $\tilde{A}(1, \alpha_1) = \alpha_1 - \overline{B} + \tilde{A}_3(1 - \mu_n)$ where $|\tilde{A}_3| \leq A_3$ and A_3 is a fundamental bound.

Take first the single-mixing case. Here if we take $A_2(1 - \mu_n) \leq (1 - \overline{B})/2$ we have $\tilde{A}(N, 1) > 0$ and we have $\tilde{A}(1, \alpha_1)$ negative for $\alpha_1 - \overline{B} < -A_3(1 - \mu_n)$ and positive for $\alpha_1 - \overline{B} > A_3(1 - \mu_n)$ implying at least one solution $\tilde{A}(1, \alpha_1) = 0$ in the interval $|\alpha_1 - \overline{B}| \leq A_3(1 - \mu_n)$ and none elsewhere. That is the first required result.

In the double mixing case we take the rectangle $|\alpha_1(1) - \overline{B}| \leq A_3(1 - \mu_n)$ and $|\alpha_1(N) - \overline{B}| \leq A_2(1 - \mu_n)$ and observe that $\tilde{A}(1, \alpha_1), \tilde{A}(N, \alpha_1)$ are not both zero outside this region. Moreover, the vector field $(\tilde{A}(1, \alpha_1(1)), \tilde{A}(N, \alpha_1(N))$ points outwards on the boundary of the rectangle. By the continuous vector field version of the Brouwer fixed point theorem there is at least one point inside the rectangle where they both vanish. $\square$